Supplemental Material:
Learning in Repeated Auctions with Budgets:
Regret Minimization and Equilibrium

Santiago R. Balseiro
Columbia University

Yonatan Gur
Stanford University

December 16, 2018

A Proofs of main results

A.1 Proof of Theorem 1

We show that for any \( \gamma < \tilde{\nu}_k/\rho_k \) no admissible bidding strategy (including randomized ones) can guarantee asymptotic \( \gamma \)-competitiveness. In the proof we use Yao’s Principle, according to which in order analyze the worst-case performance of randomized algorithms, it suffices to analyze the expected performance of deterministic algorithms given distribution over inputs.

Preliminaries. In \(^2\) we denoted by \( \mathcal{B}_k \) the class of admissible bidding strategies defined by the mappings \( \{ b^\beta_{k,t} : t = 1, \ldots, T \} \) together with the distribution \( \mathbb{P}_y \), and the target expenditure rate \( \rho_k \). We now denote by \( \tilde{\mathcal{B}}_k \subset \mathcal{B}_k \) the subclass of deterministic admissible strategies, defined by adjusting the histories to be \( \tilde{\mathcal{H}}_{k,t} := \sigma \left( \{ v_{k,\tau}, b_{k,\tau}, z_{k,\tau}, u_{k,\tau} \}_{\tau=1}^{t-1}, v_{k,t} \right) \) for any \( t \geq 2 \), with \( \tilde{\mathcal{H}}_{k,1} := \sigma (v_{k,1}) \). Then, \( \tilde{\mathcal{B}}_k \) is the subclass of bidding strategies \( \beta \in \mathcal{B}_k \) such that \( b^\beta_{k,t} \) is measurable with respect to the filtration \( \tilde{\mathcal{H}}_{k,t} \) for each \( t \in \{1, \ldots, T\} \).

To simplify notation we now drop the dependence on \( k \). Given a target expenditure rate \( \rho \), strategy \( \beta \), vector of values \( v \), vector of highest competing bids \( d \), and a constant \( \gamma \geq 1 \), we define:

\[
R^\beta_{\gamma}(v; d) = \frac{1}{T} \left( \pi^H(v; d) - \gamma \pi^\beta(v; d) \right).
\]
We next adapt Yao’s principle (Yao, 1977) to our setting to establish a connection between the expected performance of any randomized strategy and the expected performance of the best deterministic strategy under some distribution over sequences of valuations.

**Lemma A.1. (Yao’s principle)** Let $\mathbb{E}_d[-]$ denote expectation with respect to some distribution over a set of competing bid sequences $\{d^1, \ldots, d^m\} \in \mathbb{R}_+^{T \times m}$. Then, for any vector of valuations $v' \in \mathbb{R}_+^T$ and for any bidding strategy $\beta \in \mathcal{B}$,

$$\sup_{v \in [0, \bar{v}]^T} \mathbb{E}_d[\beta(v; d)] \geq \inf_{\beta \in \mathcal{B}} \mathbb{E}_d[\beta(v'; d)],$$

Lemma A.1 is an adaptation of Theorem 8.3 in Borodin and El-Yaniv (1998); For completeness we provide a proof for this Lemma in Appendix B. By Lemma A.1, to bound the worst-case loss of any admissible strategy (deterministic or not) relative to the best response in hindsight, it suffices to analyze the expected loss of deterministic strategies relative to the same benchmark, where competing bids are drawn from a carefully selected distribution.

**Worst-case instance.** Fix $T \geq \bar{v}/\rho$ and a target expenditure rate $0 < \rho \leq \bar{v}$. Suppose that the advertiser is a priori informed that the vector of best competitors’ bids is

$$\bar{v} = \left(\bar{v}, \ldots, \bar{v}, \bar{v}, \ldots, \bar{v}\right),$$

where we decompose the sequence in $m := \lceil \bar{v}/\rho \rceil$ batches of length $\lceil T/m \rceil$, and $T_0 := T - m \lfloor T/m \rfloor$ auctions are added to the end of the sequence to handle cases when $T$ is not divisible by $m$. Suppose that the advertiser also knows that the sequence of competing bids $d$ is selected randomly according to a discrete distribution $p = \{p_1, \ldots, p_m\}$ over the set $\mathcal{D} = \{d^1, d^2, \ldots, d^m\} \in [0, \bar{v}]^{T \times m}$, where:
$$d^1 = \left( \frac{d_1}{T/m} \text{ auctions}, \frac{d_2}{T/m} \text{ auctions}, \ldots, \frac{d_{m-1}}{T/m} \text{ auctions}, \frac{d_m}{T/m} \text{ auctions}, \bar{v}, \ldots, \bar{v} \right)$$

$$d^2 = \left( \frac{d_1}{T/m} \text{ auctions}, \frac{d_2}{T/m} \text{ auctions}, \ldots, \frac{d_{m-1}}{T/m} \text{ auctions}, \frac{d_m}{T/m} \text{ auctions}, \bar{v}, \bar{v}, \ldots, \bar{v} \right)$$

$$\vdots$$

$$d^{m-1} = \left( \frac{d_1}{T/m} \text{ auctions}, \frac{d_2}{T/m} \text{ auctions}, \ldots, \frac{d_{m-1}}{T/m} \text{ auctions}, \frac{d_m}{T/m} \text{ auctions}, \bar{v}, \bar{v}, \ldots, \bar{v} \right)$$

$$d^m = \left( \frac{d_1}{T/m} \text{ auctions}, \bar{v}, \ldots, \bar{v}, \bar{v}, \ldots, \bar{v} \right)$$

with $d_j = \bar{v}(1 - \varepsilon^{m-j+1})$ for every $i \in \{1, \ldots, m\}$. We use indices $i \in \{1, \ldots, m\}$ to refer to competing bid sequences and indices $j \in \{1, \ldots, m\}$ to refer to batches within a sequence (other than the last batch which always has zero utility). The parameter $\varepsilon$ is such that $\varepsilon \in (0,1]$; the precise value will be determined later on.

Sequence $d^1$ represents a case where competing bids gradually decrease throughout the campaign horizon (except the last $T_0$ auctions); sequences $d^2, \ldots, d^m$ follow essentially the same structure, but introduce the risk of net utilities going down to zero at different time points. Thus, the feasible set of competing bids present the advertiser with the following tradeoff: early auctions introduce return per unit of budget that is certain, but low. Later auctions introduce return per unit of budget that may be higher, but uncertain because the net utility (value minus payment) may diminish to zero. In the rest of the proof, the parameters of this instance are tuned to maximize the worst-case loss that must incurred due to this tradeoff by any admissible bidding strategy.

**Useful subclass of deterministic strategies.** As we next show, under the structure of the worst-case instance described above, it suffices to analyze strategies that determine before the beginning of the campaign how many auctions to win at different stages of the campaign horizon. Define the set $\mathcal{X} := \left\{ x \in \{0, \ldots, \lfloor T/m \rfloor \}^m : \sum_{j=1}^{m} d_j x_j \leq B \right\}$. Given $x \in \mathcal{X}$, we denote by $\beta^x \in \mathcal{B}$ a strategy that for each $j \in \{1, \ldots, m\}$ bids $d_j$ in the first $x_j$ auctions of batch $j$. When competing bids are $d_j$ in the $j$’th batch, this strategy is guaranteed to win $x_j$ auctions.

**Lemma A.2.** For any deterministic bidding strategy $\beta \in \mathcal{B}$ there exists a vector $x \in \mathcal{X}$ such that for any $d \in \mathcal{D}$ one has $\pi^{\beta^x}(\bar{v}, d) = \pi^{\beta}(\bar{v}, d)$.

Lemma A.2 implies that under the structure at hand, the minimal loss that is achievable by a
deterministic strategy relative to the hindsight solution is attained within the set $\mathcal{X}$. The proof of Lemma A.2 follows because every competing bid sequence $d^i$ is identical to $d^1$, thus indistinguishable, up to time $\tau^i$, the first time at which utilities go down to zero in competing bid sequence $d^i$. Therefore, the bids of any deterministic strategy coincide up to time $\tau^i$ under competing bid sequences $d^i$ and $d^1$. Because utilities after time $\tau^i$ are zero, all bids under competing bid sequence $d^i$ after time $\tau^i$ are irrelevant. Since all items in a batch have competing bid, it suffices to look at the number of auctions won by the strategy in each batch when the vector of competing bids is $d^1$.

**Analysis.** Fix $x \in \mathcal{X}$ and consider the strategy $\beta^x$. Define the net utilities matrix $U \in \mathbb{R}^{m \times m}$:

$$U = \begin{bmatrix}
  u_1 & u_2 & \ldots & u_{m-1} & u_m \\
  u_1 & u_2 & \ldots & u_{m-1} & 0 \\
  \vdots & \vdots & & \vdots & \\
  u_1 & u_2 & 0 & \ldots & 0 \\
  u_1 & 0 & 0 & \ldots & 0
\end{bmatrix}$$

with $u_j = \bar{v} - d_j = \bar{v} \varepsilon^{m-j+1}$. The matrix $U$ is invertible with rows representing different sequences in $\mathcal{D}$ and columns capturing the net utility of different batches (perhaps except the last $T_0$ auctions). Define the vector $u = (u_m, u_{m-1}, \ldots, u_1)' \in \mathbb{R}^m$ as the net utility of the most profitable batch of each sequence in $\mathcal{D}$. Define the following distribution over sequences of values:

$$p' = \frac{e' U^{-1}}{e' U^{-1} e},$$

where $e = (1, \ldots, 1)'$ and where $U^{-1}$ denotes the inverse of $U$. Some algebra shows that

$$e' U^{-1} = \frac{1}{\bar{v}} \left( \frac{1}{\varepsilon}, \frac{1}{\varepsilon^2}, \frac{1}{\varepsilon^3}, \ldots, \frac{1}{\varepsilon^m} \right),$$

and $e' U^{-1} e = 1/(\bar{v} \varepsilon^m)$. This implies that $p_i \geq 0$ for all $i \in \{1, \ldots, m\}$ because $\varepsilon \in (0, 1]$. Since, $\sum_{i=1}^m p_i = 1$ we have that $p$ is a valid probability distribution over inputs. Then, the expected performance-per-round of $\beta^x$ is:

$$\frac{1}{T} \mathbb{E}_d \left[ \pi^{\beta^x}_{\bar{v}} (\bar{v}; d) \right] = \frac{1}{T} p' U x = \frac{1}{T} \bar{v} \varepsilon^m e' x \overset{(a)}{\leq} \varepsilon^m \frac{\rho}{1 - \varepsilon}, \quad (A-1)$$

where (a) holds by Lemma A.2 because $B \geq \sum_{j=1}^m d_j x_j \geq d_m e' x$ because the competing bids $d_j$ are decreasing and $x_j \geq 0$, and using that $\rho = B/T$ and $d_m = \bar{v}(1 - \varepsilon)$. In addition, the expected
performance-per-round of the hindsight solution satisfies:

\[
\frac{1}{T} \mathbb{E}_d \left[ \pi^U(\bar{v}; \bar{d}) \right] \overset{(a)}{\geq} \frac{1}{T} \left[ \frac{T}{m} \right] p' u \overset{(b)}{=} \frac{1}{T} \left[ \frac{T}{m} \right] \varepsilon^m (m - \varepsilon (m - 1))
\]

\[
> \left( \frac{1}{m} - \frac{1}{T} \right) \bar{v} \varepsilon^m (m - \varepsilon (m - 1)),
\]

where: (a) holds since by the definition of the index \( m \) one has that \( \rho \geq \bar{v} / m \) and therefore \( B = T \rho \geq T \bar{v} / m \geq [T / m] \) \( d_j \) because \( d_j \leq \bar{v} \), so the hindsight solution allows winning at least the \( [T / m] \) most profitable auctions; and (b) holds since \( p' u = (e' U^{-1} u) / (e' U^{-1} e) = \bar{v} \varepsilon^m (m - \varepsilon (m - 1)) \).

Set \( \delta \in (0, 1) \) such that \( \gamma = (1 - \delta) (\bar{v} / \rho) \). Setting \( \varepsilon = \frac{\delta}{\delta + 1} \), one has for any \( T \geq 4 \lceil \bar{v} / \rho \rceil / \delta \):

\[
\mathbb{E}_d \left[ R_\gamma^{\beta x}(\bar{v}; \bar{d}) \right] \overset{(a)}{\geq} \bar{v} \varepsilon^m \left( \left( \frac{1}{m} - \frac{1}{T} \right) (m - \varepsilon (m - 1)) - \frac{1 - \delta}{1 - \varepsilon} \right)
\]

\[
= \bar{v} \varepsilon^m \left( \frac{\delta}{1 - \varepsilon} - \varepsilon \left( \frac{m - 1}{m} + \frac{1}{1 - \varepsilon} \right) \right)
\]

\[
\geq \bar{v} \varepsilon^m \left( \delta \left( 1 - \varepsilon \right) - \frac{m}{1 - \varepsilon} \right)
\]

\[
\overset{(b)}{=} \bar{v} \varepsilon^m \left( \delta \left( \frac{1}{1 - \varepsilon} \right) - \frac{m}{1 - \varepsilon} \right)
\]

\[
\overset{(c)}{=} \bar{v} \left( \frac{\delta}{\delta + 4} \right) \left( \frac{m}{2} \right) - \frac{m}{1 - \varepsilon}
\]

\[
\overset{(d)}{>\frac{\bar{v}}{4} \left( \delta + 4 \right) \delta^m + 1 > 0,
\]

where: (a) holds from (A-1) and (A-2); (b) holds by using that \( 1 / (1 - \varepsilon) \geq 1 \) because \( 0 < \varepsilon < \bar{v} \) for the first term in the parenthesis, using that \( (m - 1) / m \leq 1 \leq 1 / (1 - \varepsilon) \) in the second term, and dropping the last term because \( m \geq 2 \); (c) follows by setting \( \varepsilon = \frac{\delta}{\delta + 1} \); and (d) holds because \( \delta / 2 - m / T \geq \delta / 4 \) for any \( T \geq 4 \lceil \bar{v} / \rho \rceil / \delta \) since \( m = \lceil \bar{v} / \rho \rceil \) and \( \delta \in (0, 1) \). This establishes that for any \( \gamma < \bar{v} / \rho \) there is a constant \( C > 0 \) such that for \( T \) large enough:

\[
\inf_{\beta \in B} \sup_{\mathbf{v} \in [0, \bar{v}], \mathbf{d} \in \mathbb{R}_+^T} \mathbb{E}_d \left[ R_\gamma^x(\mathbf{v}; \mathbf{d}) \right] \overset{(a)}{\geq} \inf_{\beta \in B} \mathbb{E}_d \left[ R_\gamma^x(\mathbf{v}; \mathbf{d}) \right] \overset{(b)}{=} \inf_{x \in \mathbb{X}} \mathbb{E}_d \left[ R_\gamma^x(\mathbf{v}; \mathbf{d}) \right] \geq C,
\]

where (a) follows from Lemma A.1 and (b) holds by Lemma A.2 Therefore, no admissible bidding strategy can guarantee \( \gamma \)-competitiveness for any \( \gamma < \bar{v} / \rho \), concluding the proof. \qed
A.2 Proof of Theorem 2

To simplify notation we drop the dependence on \( k \). We consider an alternate process in which multipliers and bids continue being updated after the budget is depleted, but no reward is accrued after budget depletion. Denote by \( \tilde{\tau}^A := \inf\{ t \geq 1 : \tilde{B}^A_t < \tilde{v} \} \) the first auction in which the bid of the advertiser is constrained by the remaining budget, and by \( \tau^A := \tilde{\tau}^A - 1 \) the last period in which bids are unconstrained. The dynamics of the adaptive pacing strategy are:

\[
\begin{align*}
\mu_{t+1} &= P_{[0,\bar{\mu}]}(\mu_t - \epsilon (\rho - d_t x^A_t)) \\
x^A_t &= 1\{v_t - d_t \geq \mu_t d_t\} \\
\tilde{B}^A_{t+1} &= \tilde{B}^A_t - d_t x^A_t.
\end{align*}
\]

Given the sequence of competing bids \( d \) we denote by \( z^A_t = d_t x^A_t \) the expenditure at time \( t \) under the adaptive pacing strategy. Because the auction is won only if \( v_t - d_t \geq \mu_t d_t \), the expenditure satisfies \( z^A_t \leq v_t/(1 + \mu_t) \leq \bar{v} \).

**Step 1: controlling the stopping time \( \tilde{\tau}^A \).** We first show that the adaptive pacing strategy does not run out of budget too early, that is, \( \tau^A \) is close to \( T \). From the update rule of the dual variables one has for every \( t \leq \tau^A \),

\[
\begin{align*}
\mu_{t+1} &= P_{[0,\bar{\mu}]}(\mu_t + \epsilon (z^A_t - \rho)) = \mu_t + \epsilon (z^A_t - \rho) - P_t,
\end{align*}
\]

where we define \( P_t := \mu_t + \epsilon (z^A_t - \rho) - P_{[0,\bar{\mu}]}(\mu_t + \epsilon (z^A_t - \rho)) \) as the projection error. Reordering terms and summing up to period \( \tau^A \) one has

\[
\sum_{t=1}^{\tau^A} (z^A_t - \rho) = \sum_{t=1}^{\tau^A} \frac{1}{\epsilon} (\mu_{t+1} - \mu_t) + \sum_{t=1}^{\tau^A} P_t/\epsilon, \tag{A-3}
\]

We next bound each term independently. For left-hand side of \((A-3)\) we have

\[
\sum_{t=1}^{\tau^A} (z^A_t - \rho) \overset{(a)}{=} B - \tilde{B}^A_{\tau^A+1} - \rho \tau^A \overset{(b)}{\geq} \rho(T - \tau^A) - \tilde{v},
\]

where \( (a) \) holds since \( \tilde{B}^A_{\tau^A+1} = B - \sum_{t=1}^{\tau^A} z^A_t \) and \( (b) \) uses \( \tilde{B}^A_{\tau^A+1} \leq \tilde{v} \) and \( \rho = B/T \). For the first term in the right-hand side of \((A-3)\) we have
\[
\sum_{t=1}^{\tau^A} \frac{1}{\epsilon} (\mu_{t+1} - \mu_t) = \frac{\mu_{\tau^A+1}^A}{\epsilon} - \frac{\mu_1^A}{\epsilon} \leq \frac{\tilde{\mu}}{\epsilon},
\]

where the inequality follows because \( \mu_t \in [0, \tilde{\mu}] \). We next bound the second term in the right-hand side of (A-3). The projection error satisfies

\[
P_t \leq P_t^+ = (\mu_t + \epsilon (z_t^A - \rho) - P_{[0, \tilde{\mu}]}) (\mu_t + \epsilon (z_t^A - \rho))
\]

\[
\overset{(a)}{=} (\mu_t + \epsilon (z_t^A - \rho) - \tilde{\mu}) \mathbf{1} \{ \mu_t + \epsilon (z_t^A - \rho) > \tilde{\mu} \}
\]

\[
\overset{(b)}{\leq} \epsilon \bar{v} \mathbf{1} \{ \mu_t + \epsilon (z_t^A - \rho) > \tilde{\mu} \}
\]

where (a) holds since the projection error is positive only if \( \mu_t + \epsilon (z_t^A - \rho) > \tilde{\mu} \), and (b) holds since \( \mu_t \leq \tilde{\mu} \) and since the deviation is bounded by \( z_t^A - \rho \leq z_t^A \leq \bar{v} \) as the expenditure is at most \( \bar{v} \).

We next show that there is no positive projection error, or more formally, \( P_t^+ = 0 \) whenever \( \epsilon \bar{v} \leq 1 \). Consider the function \( f : \mathbb{R}_+ \to \mathbb{R} \) given by \( f(\mu) = \mu + (\epsilon \bar{v})/(1 + \mu) \). The function \( f(\cdot) \) is non-decreasing whenever \( \epsilon \bar{v} \leq 1 \), and therefore

\[
\mu_t + \epsilon (z_t^A - \rho) \overset{(a)}{\leq} \mu_t + \epsilon \left( \frac{\bar{v}}{1 + \mu_t} - \rho \right) = f(\mu_t) - \epsilon \rho
\]

\[
\overset{(b)}{\leq} f(\tilde{\mu}) - \frac{\epsilon \bar{v}}{1 + \tilde{\mu}} \overset{(c)}{=} \tilde{\mu},
\]

where (a) holds by \( z_t^A = d_t \mathbf{1} \{ v_t - d_t \geq \mu_t d_t \} \leq v_t / (1 + \mu_t) \leq \bar{v} / (1 + \mu_t) \) as a payment is never greater than \( v_t / (1 + \mu_t) \), and (b) holds since \( f(\cdot) \) is non-decreasing, \( \mu_t \leq \tilde{\mu} \), and \( \tilde{\mu} \geq \bar{v} / \rho - 1 \). Therefore, there is no positive projection error when \( \epsilon \bar{v} \leq 1 \). Using these inequalities in (A-3) one has

\[
T - \tau^A \leq \frac{\tilde{\mu}}{\epsilon \rho} + \frac{\bar{v}}{\rho}.
\]  

(A-4)

Let \( x \land y = \min(x, y) \). Therefore the strategy does not run out of budget very early, and one has

\[
\pi^A(v; d) \geq \sum_{t=1}^{T \land \tau^A} (v_t - d_t) x_t^A
\]

\[
\overset{(a)}{\geq} \sum_{t=1}^{T} (v_t - d_t) x_t^A - \bar{v} (T - \tau^A)^+
\]

\[
\overset{(b)}{\geq} \sum_{t=1}^{T} (v_t - d_t) x_t^A - \left( \frac{\bar{v} \tilde{\mu}}{\epsilon \rho} + \frac{\bar{v}^2}{\rho} \right),
\]  

(A-5)
where (a) follows from $0 \leq v_t - d_t \leq \bar{v}$, and (b) follows from (A-4).

**Step 2: bounding the regret.** We next bound the relative loss in terms of the potential value of auctions lost by the adaptive pacing strategy. Consider a relaxation of the hindsight problem (I) in which the decision maker is allowed to take fractional items. In this relaxation, the optimal solution is a greedy strategy that sorts items in decreasing order of the ratio $(v_t - d_t)/d_t$ and wins items until the budget is depleted (where the “overflowing” item is fractionally taken). Let $\mu^H$ be the sample-path dependent fixed threshold corresponding to the ratio of the “overflowing” item. Then all auctions won satisfy $(v_t - d_t) \geq \mu^H d_t$. We obtain an upper bound by assuming that the overflowing item is completely taken by the hindsight solution. The dynamics of this bound are:

$$
x_t^H = 1 \{v_t - d_t \geq \mu^H d_t\}
$$

$$
\tilde{B}_{t+1}^H = \tilde{B}_t^H - x_t^H z_t^H.
$$

Denote by $\tilde{\tau}^H := \inf \{t \geq 1 : \tilde{B}_t^H < 0\}$ the period in which the budget of the hindsight solution is depleted, and by $\tau^H := \tilde{\tau}^H - 1$. Then, one has:

$$
\pi^H(v; d) - \pi^A(v; d) \leq \sum_{t=1}^{T \wedge \tau^H} x_t^H(v_t - d_t) - \sum_{t=1}^{T \wedge \tau^A} x_t^A(v_t - d_t)
$$

$$
\leq \sum_{t=1}^{T \wedge \tau^H} (v_t - d_t) (x_t^H - x_t^A) + E_1
$$

(A-6)

where (a) follow from (A-5), ignoring periods after $\tau^H$. In addition, for any $t \in \{1, \ldots, T \wedge \tau^H\}$:

$$
(v_t - d_t) (x_t^H - x_t^A) \overset{(a)}{=} (v_t - d_t) (1 \{v_t - d_t \geq \mu^H d_t\} - 1 \{v_t - d_t \geq \mu^A d_t\})
$$

$$
\overset{(b)}{=} (v_t - d_t) (1 \{\mu^A d_t > v_t - d_t \geq \mu^H d_t\} - 1 \{\mu^H d_t > v_t - d_t \geq \mu^A d_t\})
$$

$$
\overset{(c)}{\leq} (v_t - d_t) 1 \{\mu^A d_t > v_t - d_t\} = (v_t - d_t) (1 - x_t^A),
$$

(A-7)

where: (a) follows the threshold structure of the hindsight solution as well as the dynamics of the adaptive pacing strategy; (b) holds since $x_t^H - x_t^A = 1$ if and only if the auction is won in the hindsight solution but not by the adaptive pacing strategy, and $x_t^H - x_t^A = -1$ if and only if the auction is won by the adaptive pacing strategy but not in the hindsight solution; and (c) follows from discarding indicators. Putting (A-6) and (A-7) together one obtains a bound on the relative
loss in terms of the potential value of the auctions lost by the adaptive pacing strategy:

\[
\pi^H(v; d) - \pi^A(v; d) \leq \sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) + E_1.
\]  

(A-8)

Step 3: bounding the potential value of lost auctions. We next bound the value of the auctions lost under the adaptive pacing strategy in terms of the value of alternative auctions won under the same strategy. We show that for all \( t = 1, \ldots, T \):

\[
(v_t - d_t) (1 - x_t^A) \leq \left( \frac{\bar{v}}{\rho} - 1 \right) (v_t - d_t) x_t^A + \frac{\bar{v}}{\rho} \mu_t (\rho - z_t^A).
\]  

(A-9)

Note that when \( d_t > \bar{v} \), \((A-9)\) simplifies to \( v_t - d_t \leq \bar{v} \mu_t \), because \( x_t^A = 0 \) and \( z_t^A = d_t x_t^A = 0 \). The inequality then holds since \( v_t - d_t < 0 \) and \( \mu_t \leq 0 \). We next prove \((A-9)\) holds also when \( d_t \leq \bar{v} \).

Using \( z_t^A = d_t \{ v_t - d_t \geq \mu_t d_t \} \) one obtains

\[
\mu_t (\rho - z_t^A) \leq \mu_t \left( \frac{\rho}{\bar{v}} d_t - z_t^A \right) = \mu_t d_t \left( \frac{\rho}{\bar{v}} - 1 \{ v_t - d_t \geq \mu_t d_t \} \right)
\]

\[
= \frac{\rho}{\bar{v}} \mu_t d_t \{ \mu_t d_t > v_t - d_t \} - \left( 1 - \frac{\rho}{\bar{v}} \right) \mu_t d_t \{ v_t - d_t \geq \mu_t d_t \}
\]

\[
\geq \frac{\rho}{\bar{v}} (v_t - d_t) (1 - x_t^A) - \left( 1 - \frac{\rho}{\bar{v}} \right) (v_t - d_t) x_t^A,
\]

where \( (a) \) holds because \( d_t \leq \bar{v} \), and \( (b) \) holds because \( \rho \leq \bar{v} \) and using \( \mu_t d_t > v_t - d_t \) in the first term and \( \mu_t d_t \leq v_t - d_t \) in the second term. The claim follows from multiplying by \( \bar{v} / \rho \) and reordering terms. Summing \((A-9)\) over \( t = 1, \ldots, T \) implies

\[
\sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) \leq \left( \frac{\bar{v}}{\rho} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) x_t^A + \frac{\bar{v}}{\rho} \sum_{t=1}^{T} \mu_t (\rho - z_t^A).
\]  

(A-10)

We next bound the second term in \((A-10)\). The update rule of the strategy implies that for any \( \mu \in [0, \bar{\mu}] \) one has

\[
(\mu_{t+1} - \mu)^2 \leq (\mu_t - \mu - \epsilon (\rho - z_t^A))^2
\]

\[
= (\mu_t - \mu)^2 - 2\epsilon (\mu_t - \mu)(\rho - z_t^A) + \epsilon^2 (\rho - z_t^A)^2.
\]

where \( (a) \) follows from a standard contraction property of the Euclidean projection operator. Reordering terms and summing over \( t = 1, \ldots, T \) we have for \( \mu = 0 \):

9
\[
\sum_{t=1}^{T} \mu_t (\rho - z_t^A) \leq \sum_{t=1}^{T} \frac{1}{2\epsilon} \left( (\mu_t)^2 - (\mu_{t+1})^2 \right) + \frac{\epsilon}{2} (\rho - z_t^A)^2 \\
= \frac{(\mu_1)^2}{2\epsilon} - \frac{(\mu_{T+1})^2}{2\epsilon} + \sum_{t=1}^{T} \frac{\epsilon}{2} (\rho - z_t^A)^2 \\
\leq \frac{\mu^2}{2\epsilon} + \frac{\bar{v}^2}{2} T \epsilon ,
\]  
(A-11)

where (a) follows from telescoping the sum, and (b) follows from \( \mu_t \in [0, \bar{\mu}] \) together with \( \rho, z_t^A \in [0, \bar{v}] \). Combining (A-10) and (A-11) implies

\[
\sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) \leq \left( \frac{\bar{v}}{\rho} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) x_t^A + \left( \frac{\bar{v}^2}{2\rho} \frac{1}{\epsilon} + \frac{\bar{v}^3}{2\rho} T \epsilon \right). \tag{A-12}
\]

**Step 4: putting everything together.** Using the bound on regret derived in (A-8) we have:

\[
\pi^H (v; d) - \pi^\Lambda (v; d) \leq \sum_{t=1}^{T_N^H} (v_t - d_t) (1 - x_t^A) + E_1
\leq \sum_{t=1}^{T} (v_t - d_t) (1 - x_t^A) + E_1
\leq \left( \frac{\bar{v}}{\rho} - 1 \right) \sum_{t=1}^{T} (v_t - d_t) x_t^A + E_1 + E_2
\leq \left( \frac{\bar{v}}{\rho} - 1 \right) \pi^\Lambda (v; d) + \frac{\bar{v}}{\rho} E_1 + E_2 ,
\]

where: (a) follows from adding all (non-negative) terms after \( \tau^H \) and since \( d_t \leq v_t \) for all \( t = 1, \ldots, T \); (b) follows from (A-12); and (c) follows from (A-5). Reordering terms we obtain that

\[
\pi^H (v; d) - \frac{\bar{v}}{\rho} \pi^\Lambda (v; d) \leq \frac{\bar{v}}{\rho} E_1 + E_2 = \frac{\bar{v}^3}{\rho^2} + \left( \frac{\bar{v}^2 \bar{\mu}}{\rho^2} + \frac{\bar{v}^3}{2\rho} \right) \frac{1}{\epsilon} + \frac{\bar{v}^3}{2\rho} T \epsilon ,
\]

and the result follows.

\[\square\]

**A.3 Proof of Theorem 3**

To simplify notation we drop the dependence on \( k \). We prove the result in five steps. We first upper bound the expected performance in hindsight using the optimal dual objective value \( \Psi(\mu^*) \). Then, we lower bound the performance of an adaptive pacing strategy by discarding the utility of
all auctions after budget depletion. We perform a second-order expansion of the expected utility per auction around $\mu^*$ to obtain a lower quadratic envelope involving $\Psi(\mu^*)/T$ as the zeroth-order term, the absolute mean error as the first-order term, and the mean squared error as the second-order term. The next step involves upper bounding the absolute mean errors and the mean squared errors incurred throughout the horizon. We conclude by combining these steps and using that the time of budget depletion under the strategy is “close” to $T$.

**Step 1: upper bound on the performance in hindsight.** Taking expectations in equation (4) and using Jensen’s inequality together values and competing bids being i.i.d, one obtains that

$$E_{v,d}[\pi^H(v;d)] \leq \inf_{\mu \geq 0} \Psi(\mu). \quad (A-13)$$

In the remainder of the proof, we lower bound $E_{v,d}[\pi^A(v,d)]$ in terms of $\Psi(\mu^*)$, where $\mu^*$ minimizes the dual function $\Psi(\mu) = T \left( E_{v,d}[(v - (1 + \mu)d^+] + \mu \rho \right)$. Since the dual function is differentiable, the Karush-Kuhn-Tucker optimality condition implies that $\mu^*$ satisfies the complementary condition

$$\mu^* \geq 0 \perp G(\mu^*) \leq \rho, \quad (A-14)$$

where $\perp$ denotes a complementarity condition between the multiplier and the expenditure, that is, at least one condition must hold with equality. Here we used that $\Psi'(\mu) = \rho - G(\mu)$, where $\Psi(\mu) = \Psi(\mu)/T$ and the expenditure function is $G(\mu) := E_{v,d}[1\{ (1 + \mu)d \leq v \} d]$.

**Step 2: lower bound on adaptive pacing’s performance.** We next lower bound the performance of the strategy. We consider an alternate framework in which the advertiser is allowed to bid even after budget depletion. Let $\tilde{B}_t = B - \sum_{s=1}^{t-1} z_s$ denote the advertiser’s remaining budget at the beginning of period $t$ in the alternate framework. Let $\tau = \inf\{ t \geq 1 : \tilde{B}_{t+1} < \bar{v} \}$ be the last period in which the remaining budget is larger than $\bar{v}$. Since $v/(1 + \mu) \leq \bar{v}$ for any $v \in [0, \bar{v}]$ and $\mu \geq 0$, for any period $t \leq \tau$ the bids of the advertiser are $b_t^A = v_t/(1 + \mu_t)$. Therefore the performance in both the original and alternate frameworks coincide up to time $\tau$, and therefore

$$E_{v,d}[\pi^A(v,d)] \overset{(a)}{\geq} E \left[ \sum_{t=1}^{\tau \wedge T} u_t \right] \overset{(b)}{\geq} E \left[ \sum_{t=1}^T u_t \right] - \bar{v} E \left[ (T - \tau)^+ \right], \quad (A-15)$$

where $(a)$ follows from discarding all auctions after the time the advertiser runs out of budget; and $(b)$ follows from $0 \leq u_t \leq \bar{v}$. The second term can be bounded by (A-4) because $\epsilon \bar{v} \leq 1$ from
Assumption 1 and $\bar{\mu} \geq \bar{v}/\rho - 1$. We next bound the first term.

**Step 3: lower bound on utility-per-auction.** A key step in the proof is to show that the utility per auction collected by the advertiser is “close” to $\Psi(\mu^*)$. Using the structure of the strategy, the utility of advertiser $k$ from the $t^{th}$ auction can be written as

$$u_t = 1\{d_t(1 + \mu_t) \leq v_t\}(v_t - d_t) = (v_t - (1 + \mu_t)d_t)^+ + \rho \mu_t + \mu_t(z_t - \rho).$$

Taking expectations conditional on the current multiplier $\mu_t$ we obtain

$$\mathbb{E}[u_t | \mu_t] = \Psi(\mu_t) + \mu_t (G(\mu_t) - \rho),$$

where the equality follows from the linearity of expectation and since $v_t$ and $d_t$ are independent of the multiplier $\mu_t$. Let $U(\mu) = \Psi(\mu) + \mu (G(\mu) - \rho)$ be the expected utility per auction when the multiplier is $\mu$. From the complementary slackness conditions (A-14) we have that $U(\mu^*) = \Psi(\mu^*)$.

Thrice differentiability of the dual function implies that $U(\mu)$ is twice differentiable with derivatives $U'(\mu) = \mu G'(\mu)$ and $U''(\mu) = \mu G''(\mu) + G'(\mu)$ because $\Psi'(\mu) = \rho - G(\mu)$ and the product and summation of differentiable functions is differentiable. Moreover, because the derivatives of $\Psi(\mu)$ are bounded, there exists $G' > 0$ and $G'' > 0$ such that $|G'(\mu)| \leq G'$ and $|G''(\mu)| \leq G''$, respectively.

Performing a second-order expansion around $\mu^*$ and taking expectations we obtain that

$$\mathbb{E}[u_t] = \Psi(\mu^*) + \mu^* G'(\mu^*) \mathbb{E}[\mu_t - \mu^*] + \mathbb{E} \left[ \frac{U''(\xi)}{2} (\mu_t - \mu^*)^2 \right]$$

$$\geq \Psi(\mu^*) - G' \mu^* \mathbb{E}[\mu_t - \mu^*] - \frac{\mu G''}{2} \mathbb{E}[\mu_t - \mu^*]^2,$$

where the first equation follows from Taylor’s Theorem for some $\xi$ between $\mu_t$ and $\mu^*$, and the inequality follows from the bounds on the derivatives of the expenditure function. We next upper bound the mean squared error $\delta_t$ and the absolute mean error $r_t$.

**Step 4: bound on total expected errors.** Let $\delta_t = \mathbb{E}[((\mu_t - \mu^*)^2]$ be the mean squared error. Note that the expenditure function $G$ is $\lambda$-strongly monotone since the dual function is strongly convex with parameter $\lambda$. Because the step size satisfies $2\lambda \epsilon \leq 1$, equation (A-21) implies that

$$\delta_t \leq \bar{\mu}^2 (1 - 2\lambda \epsilon)^{t-1} + \frac{\bar{v}^2}{2\lambda \epsilon}.$$
Thus, the total mean squared error is bounded by

\[
\sum_{t=1}^{T} \delta_t \leq \frac{\bar{v}^2 T}{2\lambda} + \bar{\mu}^2 \sum_{t=1}^{T} (1 - 2\lambda \epsilon)^{t-1} \leq \frac{\bar{v}^2}{2\lambda} T \epsilon + \bar{\mu}^2 \frac{1}{2\lambda} \epsilon,
\]

where the last inequality follows from \(\sum_{t=1}^{T} (1 - 2\lambda \epsilon)^{t-1} \leq \sum_{t=0}^{\infty} (1 - 2\lambda \epsilon)^{t} = \frac{1}{2\lambda \epsilon}\).

Let \(r_t = \mu^* \| \mathbb{E} [\mu_{t+1} - \mu^*] \|\) be the absolute mean error (times the optimal multiplier). We proceed under the assumption that \(\mu^* > \epsilon \rho\). Otherwise, we can use bound \(r_t \leq \epsilon \rho \bar{\mu}\), which is sufficient for our results. From the update rule of the dual variables one has

\[
\mu_{t+1} = P_{[0,\bar{\mu}]}(\mu_t + \epsilon (z_t - \rho)) = \mu_t + \epsilon (z_t - \rho) - P_t,
\]

where we define \(P_t := \mu_t + \epsilon (z_t - \rho) - P_{[0,\bar{\mu}]}(\mu_t + \epsilon (z_t - \rho))\) as the projection error. Subtracting \(\mu^*\), multiplying by \(\mu^*\) and taking expectations we obtain that

\[
\mu^* \mathbb{E} [\mu_{t+1} - \mu^*] \overset{(a)}{=} \mu^* \mathbb{E} [\mu_t - \mu^*] + \epsilon \mu^* (\mathbb{E} [G(\mu_t)] - G(\mu^*)) - \mu^* \mathbb{E} [P_t]
\]

\[
\overset{(b)}{=} (1 + \epsilon G'(\mu^*)) \mu^* \mathbb{E} [\mu_t - \mu^*] + \frac{G''(\xi)}{2} \epsilon \mu^* \mathbb{E} [(\mu_t - \mu^*)^2] - \mu^* \mathbb{E} [P_t],
\]

where (a) follows because values \(v_t\) and competing bids \(d_t\) are independent of the multiplier \(\mu_t\), and \(\mu^* (G(\mu^*) - \rho) = 0\) from the complementary slackness conditions \((A-14)\); and (b) follows from Taylor’s Theorem for some \(\xi\) between \(\mu^*\) and \(\mu_t\) because \(G\) is twice differentiable. We next bound the third term in the right-hand side of \((A-17)\). The projection error satisfies

\[
P_t \overset{(a)}{\geq} (\mu_t + \epsilon (z_t - \rho)) 1 \{ \mu_t + \epsilon (z_t - \rho) < 0 \} \overset{(b)}{\geq} -\epsilon \rho 1 \{ \mu_t < \epsilon \rho \},
\]

where (a) holds as the projection error is negative only if \(\mu_t + \epsilon (z_t - \rho) < 0\), and (b) holds by \(\mu_t \geq 0\) and the expenditure being non-negative. Taking expectations, one has by Markov’s inequality that

\[
-\mathbb{E} [P_t] \leq \epsilon \rho \mathbb{P} \{ \mu_t < \epsilon \rho \} \leq \epsilon \rho \mathbb{P} \{ |\mu_t - \mu^*| \geq |\mu^* - \epsilon \rho| \} \leq \frac{\epsilon \rho \delta_t}{(\mu^* - \epsilon \rho)^2}.
\]

Taking absolute values in \((A-17)\) and invoking the triangle inequality we conclude that

\[
r_{t+1} \leq (1 - \lambda \epsilon) r_t + \epsilon \left( \frac{\bar{\mu} \rho}{(\mu^* - \epsilon \rho)^2} + \frac{\bar{\mu} G''}{2} \right) \delta_t,
\]

13
where we used that \( \mu^* \leq \bar{\mu} \), \( \sup_{\mu \in [0, \bar{\mu}]} G'(\mu) \leq -\lambda \), and \( \sup_{\mu \in [0, \bar{\mu}]} G''(\mu) \leq \bar{G}'' \) as the dual function has bounded derivatives. By Lemma \[C.4\] with \( a = \lambda \epsilon \leq 1 \) and \( b_t = \epsilon R \delta_t \) together with \( \mu^* \leq \bar{\mu} \) and \( r_1 \leq \bar{\mu}^2 \), one has

\[
 r_t \leq \bar{\mu}^2 (1 - \lambda \epsilon)^{t-1} + \epsilon R \sum_{s=1}^{t-1} (1 - \lambda \epsilon)^{t-1-s} \delta_s.
\]

Thus, the total absolute mean error is bounded by

\[
\sum_{t=1}^{T} r_t \leq \bar{\mu}^2 \sum_{t=1}^{T} (1 - \lambda \epsilon)^{t-1} + \epsilon R \sum_{t=1}^{T-1} \sum_{s=1}^{t-1} (1 - \lambda \epsilon)^{t-1-s} \delta_s
\]

\[
= \bar{\mu}^2 \sum_{t=0}^{T-1} (1 - \lambda \epsilon)^{t} + \epsilon R \sum_{s=1}^{T-1} \delta_s \sum_{t=0}^{T-1-s} (1 - \lambda \epsilon)^{t}
\]

\[
\leq \frac{\bar{\mu}^2}{\lambda \epsilon} + \frac{R}{\lambda} \sum_{s=1}^{T} \delta_s,
\]

where the equality follows from changing the order of the summation and the last inequality because \( \sum_{t=0}^{T} (1 - \lambda \epsilon)^{t} \leq \frac{1}{\lambda \epsilon} \) for all \( s \geq 0 \).

**Step 5: putting everything together.** Combining equations \[(A-13), (A-15), (A-4),\] summing \[(A-16)\] over \( t = 1, \ldots, T \), and using \[(A-17)\] and \[(A-19)\] we conclude that there exist constants \( C_1, C_2, C_3 > 0 \) such that

\[
\mathbb{E}_{v, d} [\pi^H(v; d) - \pi^A(v, d)] \leq C_1 + \frac{C_2}{\epsilon} + C_3 T \epsilon.
\]

Thus the regret converges to zero under Assumption \[1\] and setting the step size to \( \epsilon \sim T^{-1/2} \) yields a convergence rate of \( T^{-1/2} \). This concludes the proof. \( \square \)

**A.4 Proof of Theorem 4**

We prove the result by first bounding the mean squared error when all bids are unconstrained, i.e., \( \hat{B}_{k,t+1} \geq \hat{v}_k \) for every advertiser \( k \). We then argue that budgets are not depleted too early when the strategy is followed by all advertisers. We conclude by combining these results to upper bound the time average mean squared error.

**Step 1.** At a high level, this step adapts standard stochastic approximation results (see, e.g., Nemirovski et al. 2009) to the expenditure observations and to accommodate the possibility of different step sizes across agents. Fix some \( k \in \{1, \ldots, K\} \). The squared error satisfies the recursion
\[ |\mu_{k,t+1} - \mu_k^*|^2 = \left| P_{[0,\bar{\rho}_k]}(\mu_{k,t} - \epsilon_k (\rho_k - z_{k,t})) - \mu_k^* \right|^2 \]

\[ \leq |\mu_{k,t} - \mu_k^* - \epsilon_k (\rho_k - z_{k,t})|^2 \]

\[ = |\mu_{k,t} - \mu_k^*|^2 - 2\epsilon_k (\mu_{k,t} - \mu_k^*) (\rho_k - z_{k,t}) + \epsilon_k^2 |\rho_k - z_{k,t}|^2, \]

where (a) follows from a standard contraction property of the Euclidean projection operator.

Given a vector of multipliers \( \mu \), let \( G_k(\mu) := \mathbb{E}_v \left[ (1 + \mu_k) d_k \leq v_k \right] d_{k,1} \) be the expected expenditure under the second-price auction allocation rule, with \( d_k = \max_{i \neq k} \left\{ v_i / (1 + \mu_i) \right\} \) and the expectation taken w.r.t. the values \( v_k \sim F_k \). Define \( \delta_{k,t} := \mathbb{E}_v \left[ (\mu_{k,t} - \mu_k^*)^2 \mathbb{1}\{\tilde{B}_{k,t+1} \geq \bar{v}_k \forall k\} \right] \)

and \( \delta_t := \sum_{k=1}^K \delta_{k,t} \). Similarly, define \( \hat{\delta}_{k,t} := \delta_{k,t} / \epsilon_k \) and \( \hat{\delta}_t := \sum_{k=1}^K \hat{\delta}_{k,t} \). Taking expectations and dividing by \( \epsilon_k \) we obtain that

\[ \hat{\delta}_{k,t+1} \overset{(a)}{\leq} \hat{\delta}_{k,t} - 2\mathbb{E} \left[ (\mu_{k,t} - \mu_k^*) (\rho_k - z_{k,t}) \right] + \epsilon_k \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right] \]

\[ = \hat{\delta}_{k,t} - 2\mathbb{E} \left[ (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) \right] + \epsilon_k \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right], \quad (A-20) \]

where (a) holds since remaining budgets monotonically decrease with \( t \), and by conditioning on the multipliers \( \mu_t \) and using the independence of \( v_{k,t} \) from the multipliers \( \mu_t \) to obtain that \( \mathbb{E}[Z_{k,t}|\mu_t] = G_k(\mu_t) \). For the second term in (A-20) one has:

\[ (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) = (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu^*) + G_k(\mu^*) - G_k(\mu_t)) \]

\[ \geq (\mu_{k,t} - \mu_k^*) (G_k(\mu^*) - G_k(\mu_t)), \]

where the inequality follows because \( \mu_{k,t} \geq 0 \) and \( \rho_k - G_k(\mu^*) \geq 0 \) and \( \mu_k^* (\rho_k - G_k(\mu^*)) = 0 \). Summing over the different advertisers and Assumption 2 part (i) we obtain

\[ \sum_{k=1}^K (\mu_{k,t} - \mu_k^*) (\rho_k - G_k(\mu_t)) \geq \sum_{k=1}^K (\mu_{k,t} - \mu_k^*) (G_k(\mu^*) - G_k(\mu_t)) \geq \lambda \mu_t - \mu^* \|^2_2. \]

Let \( \bar{v} = \max_k \bar{v}_k \). In addition, for the third term in (A-20) we have \( \mathbb{E} \left[ |\rho_k - z_{k,t}|^2 \right] \leq \bar{v}_k^2 \leq \bar{v}^2 \), since \( \bar{v}_k \geq \rho_k \geq 0 \) and \( z_{k,t} \geq 0 \), and since the payment is at most the bid \( v_{k,t} / (1 + \mu_{k,t}) \leq v_{k,t} \leq \bar{v}_k \) because \( \mu_{k,t} \geq 0 \). Denoting \( \bar{\epsilon} = \max_{k \in \{1,\ldots,K\}} \epsilon_k \), \( \epsilon = \min_{k \in \{1,\ldots,K\}} \epsilon_k \), we conclude by summing (A-20) over \( k \) that:

\[ \hat{\delta}_{t+1} \leq \hat{\delta}_t - 2\lambda \delta_t + \bar{\epsilon} \bar{v} \overset{(a)}{\leq} (1 - 2\lambda \bar{\epsilon}) \hat{\delta}_t + K \bar{v}^2 \hat{\delta}_t, \]

15
where \((a)\) follows from \(\delta_t = \sum_{k=1}^{K} \delta_{k,t} = \sum_{k=1}^{K} \epsilon_k \hat{\delta}_{k,t} \geq \epsilon \hat{\delta}_t\) because \(\delta_{k,t} \geq 0\). Lemma C.4 with \(a = 2\lambda \epsilon \leq 1\) and \(b = \epsilon K \overline{\nu}^2\) implies that

\[
\hat{\delta}_t \leq \hat{\delta}_1 (1 - 2\lambda \epsilon)^{t-1} + \frac{K \overline{\nu}^2 \epsilon}{2\lambda \epsilon}.
\]

Using that \(\delta_t \leq \epsilon \hat{\delta}_t\) together with \(\hat{\delta}_1 \leq \delta_1/\epsilon \leq K \bar{\mu}^2/\epsilon\) because \(\mu_{k,t}, \mu_k^* \in [0, \bar{\mu}]\) and \(\bar{\mu} = \max_k \bar{\mu}_k\) we obtain that

\[
\delta_t \leq K \bar{\mu}^2 \frac{\epsilon}{\epsilon} (1 - 2\lambda \epsilon)^{t-1} + \frac{K \overline{\nu}^2 \epsilon^2}{2\lambda \epsilon}.
\]

**Step 2.** Let \(\tilde{\tau}_k\) be the first auction in which the remaining budget of advertiser \(k\) is less than \(\bar{v}_k\) (at the beginning of the auction), that is, \(\tilde{\tau}_k = \inf\{t \geq 1 : \tilde{B}_{k,t} < \bar{v}_k\}\). Let \(\tau_k := \tilde{\tau}_k - 1\) be the last period in which the remaining budget of advertiser \(k\) is greater than \(\bar{v}_k\). Let \(\tau = \min_{k=1,\ldots,K}\{\tau_k\}\). Since \(v/(1 + \mu) \leq \bar{v}_k\) for any \(v \in [0, \bar{v}_k]\) and \(\mu \geq 0\), for any period \(t \leq \tau\) the bids of all advertisers are guaranteed to be \(b_{k,t}^* = v_{k,t}/(1 + \mu_{k,t})\). Inequality \(\text{(A-4)}\) implies that for each bidder \(k\), the stopping time satisfies:

\[
T - \tau_k \leq \frac{\bar{\mu}_k}{\epsilon_k \rho_k} + \frac{\bar{v}_k}{\rho_k},
\]

and therefore, denoting \(\underline{\rho} = \min_k \rho_k\), \(\bar{\nu} = \max_k \bar{v}_k\), and \(\bar{\mu} = \max_k \bar{\mu}_k\), one has:

\[
T - \tau = T - \min_{k=1,\ldots,K}\{\tau_k\} = \max_{k=1,\ldots,K}\{T - \tau_k\} \leq \frac{\bar{\mu}}{\epsilon \underline{\rho}} + \frac{\bar{\nu}}{\underline{\rho}}.
\]

**Step 3.** Putting everything together we obtain that

\[
\sum_{t=1}^{T} \mathbb{E}_v \left[ \|\mu_t - \mu^*\|^2 \right] \overset{(a)}{=} \sum_{t=1}^{T} \mathbb{E}_v \left[ \|\mu_t - \mu^*\|^2 \mathbf{1}\{t \leq \tau\} + \|\mu_t - \mu^*\|^2 \mathbf{1}\{t > \tau\} \right]
\]

\[
\overset{(b)}{\leq} \sum_{t=1}^{T} \delta_t + K \bar{\mu}^2 \mathbb{E}_v [(T - \tau)^+]
\]

\[
\overset{(c)}{\leq} K \bar{\mu}^2 \frac{\epsilon}{\epsilon} \sum_{t=1}^{T} (1 - 2\lambda \epsilon)^{t-1} + T \frac{K \overline{\nu}^2 \epsilon^2}{2\lambda \epsilon} + K \bar{\mu}^2 \left( \frac{\bar{\mu}}{\epsilon \underline{\rho}} + \frac{\bar{\nu}}{\underline{\rho}} \right)
\]

\[
\overset{(d)}{\leq} \frac{K \bar{\mu}^2}{2\lambda \epsilon^2} + T \frac{K \overline{\nu}^2 \epsilon^2}{2\lambda \epsilon} + K \bar{\mu}^2 \left( \frac{\bar{\mu}}{\epsilon \underline{\rho}} + \frac{\bar{\nu}}{\underline{\rho}} \right),
\]

where \((a)\) follows from conditioning on the stopping time, \((b)\) follows from the definition of \(\delta_t\) and using that \(\mu_t \in [0, \bar{\mu}_k]\), \((c)\) follows from \(\text{(A-21)}\) and \(\text{(A-22)}\), and \((d)\) follows from \(\sum_{t=1}^{T} (1 - 2\lambda \epsilon)^{t-1} \leq \sum_{t=0}^{\infty} (1 - 2\lambda \epsilon)^t = \frac{1}{2\lambda \epsilon}\) because \(2\lambda \epsilon < 1\). Dividing by \(T\), the result then follows because, as \(T \to \infty\), the first and third term go to zero by Assumption B part (iii), using \(\epsilon \leq \bar{\epsilon}\); and the second term
goes to zero by Assumption 3 part (ii).

A.5 Proof of Theorem 5

In the proof we consider an alternate framework in which advertisers are also allowed to bid after depleting their budget. The main idea of the proof lies in analyzing the performance of a given advertiser, showing that its performance in the original framework (throughout its entire campaign) is close to the one it achieves in the alternate one before some advertiser runs out of budget.

Preliminaries and auxiliary results. Consider the sequence \{(z_{k,t}, u_{k,t})\}_{t \geq 1} of realized expenditures and utilities of advertiser \(k\) in the alternate framework. Then, with the competing bid given by 

\[ d_{k,t} = \max_{i \neq k} \frac{v_{i,t}}{1 + \mu_{i,t}}, \]

one has under the second-price allocation rule

\[ z_{k,t} = 1 \{ (1 + \mu_{k,t})d_{k,t} \leq v_{k,t} \} d_{k,t}, \]

and 

\[ u_{k,t} = 1 \{ (1 + \mu_{k,t})d_{k,t} \leq v_{k,t} \} (v_{k,t} - d_{k,t}). \]

Let \( \hat{B}_{k,t} = B_k - \sum_{s=1}^{t-1} z_{s,k} \) denote the evolution of the \(k\)th advertiser’s budget at the beginning of period \(t\) in the alternate framework.

Let 

\[ \tau = \inf \{ t \geq 1 : \hat{B}_{k,t+1} < \bar{v}_k \text{ for some } k = 1, \ldots, K \} \]

be the last period in which the remaining budget of all advertisers is larger than \(\bar{v}_k\). Since \(v/(1 + \mu) \leq \bar{v}_k\) for any \(v \in [0, \bar{v}_k]\) and \(\mu \geq 0\), for any period \(t \leq \tau\) the bids of all advertisers are guaranteed to be 

\[ b_{k,t} = \frac{v_{k,t}}{1 + \mu_{k,t}}. \]

Denoting \(\bar{\rho} = \min_k \rho_k\), \(\bar{v}_k = \max_k \bar{v}_k\), and \(\bar{\mu}_k = \max_k \bar{\mu}_k\), inequality \(\text{(A-22)}\) implies that

\[ T - \tau \leq \frac{\bar{\mu}}{\bar{\rho}} + \frac{\bar{v}}{\bar{\rho}}. \]

A key step in the proof involves showing that the utility-per-auction collected by the advertiser is “close” relative to \(\Psi_k(\mu^*)\). The next result bounds the performance gap of the adaptive pacing strategy relative to \(\Psi_k(\mu^*)\), in terms of the expected squared error of multiplier. Denote by \(\mu_t \in \mathbb{R}^K\) the (random) vector such that \(\mu_{i,t}\) is the multiplier used by advertiser \(i\) at time \(t\).

Lemma A.3. Let each advertiser \(k\) follow the adaptive pacing strategy \(A\) in the alternate framework where budgets are not enforced. Then, there exists a constant \(C_3 > 0\) such that:

\[ \mathbb{E}[u_{k,t}] \geq \frac{\Psi_k(\mu^*)}{T} - C_3 \left( \mathbb{E}\left[ \left\| \mu_t - \mu^* \right\|_2^2 \right] \right)^{1/2}. \]

We note that \(C_3 = (2\bar{v} + \bar{\mu} \bar{v}^2 \bar{f})K^{1/2}\). The proof of Lemma A.3 is established by writing the utility in terms of the dual function and the complementary slackness condition, and then using that the dual and functions are Lipschitz continuous as argued in Lemma C.2. The proof of this result is deferred to Appendix B.
Proving the result. Let each advertiser \( k \) follow the adaptive pacing strategy with step size \( \epsilon_k \). The performance of both the original and the alternate systems coincide until time \( \tau \), and therefore:

\[
\Pi_k^A \overset{(a)}{=} \mathbb{E} \left[ \sum_{t=1}^{\tau \wedge T} u_{k,t} \right] \geq \mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] - \bar{v}_k \mathbb{E} \left[ (T - \tau)^+ \right],
\]

where \((a)\) follows from discarding all auctions after the time some advertiser runs out of budget; and \((b)\) follows from \( 0 \leq u_{k,t} \leq \bar{v}_k \). Summing the lower bound on the expected utility-per-auction in Lemma \A.3 over \( t = 1, \ldots, T \) one has:

\[
\mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] \geq \mathbb{E} \left[ \| \mu_t - \mu^* \|_2^2 \right]^{1/2}
\]

\[
\overset{(a)}{=} \Psi_k(\mu^*) - C_3 C_1^{1/2} \left( \frac{\bar{v}}{\lambda} \right)^{1/2} \sum_{t=1}^{T} (1 - 2\lambda \epsilon)_{(t-1)/2} - C_3 C_2^{1/2} \frac{\bar{v} \epsilon}{\epsilon^{1/2}} T \quad \text{(A-23)}
\]

\[
\overset{(b)}{=} \Psi_k(\mu^*) - C_3 C_1^{1/2} \frac{\epsilon^{1/2}}{\lambda^{3/2}} - C_3 C_2^{1/2} \frac{\bar{v} \epsilon}{\epsilon^{1/2}} T. \quad \text{(A-24)}
\]

where \((a)\) follows from Theorem \[1\] and \( \sqrt{x+y} \leq \sqrt{x} + \sqrt{y} \) for \( x, y \geq 0 \), and \((b)\) follows from \( \sum_{t=1}^{T} (1 - 2\lambda \epsilon)_{(t-1)/2} \leq \sum_{t=0}^{\infty} (1 - 2\lambda \epsilon)_{t/2} = \frac{1}{1 - (1 - 2\lambda \epsilon)^{1/2}} \leq \frac{1}{\lambda \epsilon} \) because \( 1 - (1 - x)^{1/2} \geq x/2 \) for \( x \in [0, 1] \). We use the bound in \[\text{A-22}\] to bound the truncated expectation as follows:

\[
\mathbb{E} \left[ (T - \tau)^+ \right] \leq \frac{\bar{v} \mu}{\epsilon^{1/2}} + \frac{\bar{v}^2}{\rho}. \quad \text{(A-25)}
\]

Combining \[\text{A-23}\] and \[\text{A-25}\] we obtain that

\[
\Pi_k^A \geq \mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] - \bar{v}_k \mathbb{E} \left[ (T - \tau)^+ \right]
\]

\[
\geq \Psi_k(\mu^*) - \frac{C_3 C_1^{1/2} \epsilon^{1/2}}{\lambda^{3/2}} - C_3 C_2^{1/2} \frac{\bar{v} \epsilon}{\epsilon^{1/2}} T - \frac{\bar{v} \mu}{\epsilon^{1/2}} - \frac{\bar{v}^2}{\rho}. \quad \text{(A-26)}
\]

The dependence of the payoff gap on the number of advertisers \( K \) is of the same as the dependencies of the products \( C_3 C_1^{1/2} \) and \( C_3 C_2^{1/2} \), which are of order \( K \). This concludes the proof.

\[\square\]

A.6 Proof of Theorem [6]

As in the proof of Theorem [5], we consider an alternate framework in which advertisers are allowed to bid even after budget depletion, without any utility gained. The performance of a deviating
advertiser (indexed $k$) in the original framework is equal to the one in the alternate framework up to the first time some advertiser runs out of budget. As an adaptive pacing strategy does not run out of budget too early, the main idea of the proof lies in analyzing the performance of the deviating advertiser in the alternate framework. We first argue that when the number of competitors is large, the expected squared error of the competitors’ multipliers relative to the vector $\mu^*$ is small, since the impact of the deviating advertiser on its competitors is limited. We then show that the benefit of deviating to any other strategy is small when competitors’ multipliers are “close” to $\mu^*$.

**Preliminaries and auxiliary results.** Consider the sequence $\{(z_{k,t}, u_{k,t})\}_{t\geq 1}$ of realized expenditures and utilities of advertiser $k$ in the alternate framework, and let $\{b_{k,t}^\beta\}_{t\geq 1}$ be the bids of advertiser $k$. Then, with the competing bid given by $d_{k,t} = \max_{i \neq k} m_{k,t} = m_{i,t} v_{i,t} / (1 + \mu_{i,t})$, one has under the second-price allocation rule that $z_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} d_{k,t}$, and $u_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} (v_{k,t} - d_{k,t})$. The competing bid faced by an advertiser $i \neq k$ is given by $d_{i,t} = b_{k,t}^\beta \lor \max_j \min_{i \neq k} \{m_j, v_{j,t} / (1 + \mu_{j,t})\}$. The history available at time $t$ to advertiser $k$ in the model described in [5] is defined by

$$H_{k,t} := \sigma\left(\left\{m_{k,\tau}, v_{k,\tau}, b_{k,\tau}, z_{k,\tau}, u_{k,\tau}\right\}_{\tau=1}^{t-1}, m_{k,t}, v_{k,t}, y\right)$$

for any $t \geq 2$, with $H_{k,1} := \sigma(m_{k,1}, v_{k,1}, y)$. In what follows we provide few auxiliary results that we use in the proof. The complete proofs of these results are deferred to Appendix [5].

Let $\tau_i$ be the first auction in which the remaining budget of advertiser $i \neq k$ is less than $\tilde{v}_i$ (just before the auction); that is, $\tau_i = \inf\{t \geq 1 : \tilde{B}_{i,t} < \tilde{v}_i\}$, and let $\tau = \min_{i \neq k} \{\tau_i\}$. Since for each advertiser $i \neq k$ the bid satisfies $v / (1 + \mu) \leq \tilde{v}_i$ for any $v \in [0, \tilde{v}_i]$ and $\mu \geq 0$, in each period $t \leq \tau$ the bid is $b_{i,t} = v_{i,t} / (1 + \mu_{i,t})$. Inequality (A-22) implies that the stopping time satisfies

$$T - \tau \leq \frac{\bar{\mu}}{\bar{\rho}} + \frac{\bar{v}}{\bar{\rho}}$$

with $\rho = \min_{i \neq k} \rho_i$, $\bar{v} = \max_{i \neq k} \bar{v}_i$, and $\bar{\mu} = \max_{i \neq k} \bar{\mu}_i$. First, we show that the mean squared errors of the estimated multipliers at period $t$ can be bounded in terms of the minimum and maximum step sizes, the number of players and $a_k = (a_{k,i})_{i \neq k} \in [0,1]^{K-1}$. Denote by $\mu_t \in \mathbb{R}^K$ the (random) vector such that $\mu_{k,t} = \mu_k^*$ and $\mu_{i,t}$ is the multiplier used by advertiser $i \neq k$ at time $t$.

**Lemma A.4.** Suppose that Assumption [5] holds and let $\mu^*$ be the profile of multipliers defined by [5]. Let each advertiser $i \neq k$ follow the adaptive pacing strategy $A$ with $\bar{e} \leq 1 / (2\lambda)$ and suppose that advertiser $k$ uses some strategy $\beta \in B_k^{\infty}$. Then, there exist positive constants $C_1, C_2, C_3$ independent
Suppose that Assumption 2 holds and let Lemma A.5. Let each advertiser strategy relative to $\Psi$ its competitors. The next result bounds the Lagrangian utility per auction of an adaptive pacing strategy $\mu$ the vector $\mu$ under some assumptions on the step sizes and how often advertiser $k$ interacts with its competitors. The next result bounds the Lagrangian utility per auction of an adaptive pacing strategy relative to $\Psi_k(\mu^*)$, in terms of the multipliers’ expected squared error.

**Lemma A.5.** Suppose that Assumption 2 holds and let $\mu^*$ be the profile of multipliers defined by (5). Let each advertiser $i \neq k$ follow the adaptive pacing strategy $A$. Then, there exists a constant $C_4 > 0$ such that for any $t \in \{1, \ldots, T\}$:

$$\mathbb{E} [u_{k,t} - \mu^*_k z_{k,t} + \rho_k] \leq \frac{\Psi_k(\mu^*)}{T} + C_4 \|a_k\|_2 \cdot \left(\mathbb{E} \|\mu_t - \mu^*\|_2^2\right)^{1/2}.$$ 

**Proving the result.** Let $\mu^*$ be the profile of multipliers defined by (5). Let each advertiser $i \neq k$ follow the adaptive pacing strategy with step size $\epsilon_k$, except advertiser $k$ who implements a strategy $\beta \in B_{k^2}$. Both the original and the alternate systems coincide up to time $\tau$, and thus:

$$H_k^{\beta, A, k} \leq \frac{\mathbb{E} \left( \sum_{t=1}^{T} u_{k,t} \right)}{T} + \bar{v}_k \mathbb{E} \left( (T - \tau)^+ \right) \leq \sum_{t=1}^{T} \mathbb{E} \left[ u_{k,t} - \mu^*_k z_{k,t} + \rho_k \right] + \bar{v}_k \mathbb{E} \left[ (T - \tau)^+ \right],$$

where (a) follows from adding all auctions after time $\tau$ in the alternate system, and using that the utility of each auction in the original system satisfy $0 \leq u_{k,t} \leq \bar{v}_k$; and (b) follows from adding the constraint $\sum_{t=1}^{T} z_{k,t} \leq B_k$ to the objective with a Lagrange multiplier $\mu^*_k$ because the strategy $\beta$ is budget-feasible in the alternate framework. Summing the lower bound on the expected utility-per-auction in Lemma A.5 over $t = 1, \ldots, T$ one has:

$$\sum_{t=1}^{T} \mathbb{E} [u_{k,t} - \mu^*_k z_{k,t} + \rho_k] \leq \Psi_k(\mu^*) + C_4 \|a_k\|_2 \sum_{t=1}^{T} \left( \mathbb{E} \|\mu_t - \mu^*\|_2^2 \right)^{1/2}.$$ 

The sum in the second term can be upper bounded by

$$\sum_{t=1}^{T} \left( \mathbb{E} \|\mu_t - \mu^*\|_2^2 \right)^{1/2} \leq C_1^{1/2} K^{1/2} \left( \frac{\bar{\epsilon}}{\bar{\xi}} \right)^{1/2} \sum_{t=1}^{T} (1 - \lambda_{t})^{(t-1)/2} + C_2^{1/2} K^{1/2} \frac{\bar{\epsilon}}{\bar{\xi}^{1/2}} T + C_3^{1/2} \|a_k\|_2^2 \frac{\bar{\xi}}{\bar{\xi} T},$$

(b) $\leq 2 C_1^{1/2} K^{1/2} \frac{\bar{\epsilon}^{1/2}}{\bar{\xi}^{3/2}} + C_2^{1/2} K^{1/2} \frac{\bar{\epsilon}}{\bar{\xi}^{1/2}} T + C_3^{1/2} \|a_k\|_2^2 \frac{\bar{\xi}}{\bar{\xi} T},$
where (a) follows from Lemma A.4 and $\sqrt{x + y + z} \leq \sqrt{x} + \sqrt{y} + \sqrt{z}$ for $x, y, z \geq 0$, and (b) follows from $\sum_{t=1}^{T} (1 - \lambda t)^{t-1/2} \leq \sum_{t=0}^{\infty} (1 - \lambda t)^{t/2} = \frac{1}{1 - (1 - \lambda^{1/2})x}$ because $1 - (1 - x)^{1/2} \geq x/2$ for $x \in [0, 1]$. Therefore, we obtain that

$$\sum_{t=1}^{T} \mathbb{E} [u_{k,t} - \mu_{k,z_{k,t}} + \rho_{k}] \leq \Psi_{k}(\mu^{*}) + \frac{2C_{4}C_{4}^{1/2}}{\lambda} \|a_{k}\|_{\text{i}}^{K^{1/2}} \frac{\epsilon^{1/2}}{\xi^{3/2}}$$

$$+ C_{4}C_{2}^{1/2} \|a_{k}\|_{\text{i}}^{K^{1/2}} \frac{\epsilon}{\xi^{1/2}} T + C_{4}C_{3}^{1/2} \|a_{k}\|^{2} \frac{\epsilon}{\xi^{1/2}} T.$$

(A-28)

We use the bound in (A-27) to bound the truncated expectation as follows:

$$\mathbb{E} [(T - \tau)^{+}] \leq \frac{\bar{\mu}}{\xi \rho} + \bar{\nu}.$$

(A-29)

Combining (A-28) and (A-29) one obtains that

$$\Pi_{k}^{\beta, A-k} \leq \mathbb{E} \left[ \sum_{t=1}^{T} u_{k,t} \right] + \bar{v}_{k} \mathbb{E} [ (T - \tau)^{+}]$$

$$\leq \Psi_{k}(\mu^{*}) + \frac{2C_{4}C_{4}^{1/2}}{\lambda} \|a_{k}\|_{\text{i}}^{K^{1/2}} \frac{\epsilon^{1/2}}{\xi^{3/2}}$$

$$+ C_{4}C_{2}^{1/2} \|a_{k}\|_{\text{i}}^{K^{1/2}} \frac{\epsilon}{\xi^{1/2}} T + C_{4}C_{3}^{1/2} \|a_{k}\|^{2} \frac{\epsilon}{\xi^{1/2}} T + \frac{\bar{\mu} \bar{v}}{\xi \rho} + \bar{\nu}^{2}.$$

The result follows from combining the last inequality with (A-26).

B Proofs of key lemmas

Proof of Lemma A.1 To simplify notation we drop the dependence on $k$. Let $\mathbb{P}\{d^{i}\}$ be the probability that competing bid sequence $d^{i}$ is chosen, for $i = 1, \ldots, m$, and let $\beta$ be a (potentially randomized) strategy. We have:

$$\sup_{v \in [0, \hat{\nu}]^{T}} \mathbb{E}\beta \left[ R_{\gamma}^{\beta}(v; d) \right] \stackrel{(a)}{=} \sum_{i=1}^{m} \mathbb{P}\{d^{i}\} \sup_{v \in [0, \hat{\nu}]^{T}} \mathbb{E}\beta \left[ R_{\gamma}^{\beta}(v; d) \right] \geq \sum_{i=1}^{m} \mathbb{P}\{d^{i}\} \mathbb{E}\beta \left[ R_{\gamma}^{\beta}(v'; d^{i}) \right]$$

$$\equiv \mathbb{E}\beta \left[ \sum_{i=1}^{m} \mathbb{P}\{d^{i}\} R_{\gamma}^{\beta}(v'; d^{i}) \right] \geq \inf_{\beta \in B} \left[ \sum_{i=1}^{m} \mathbb{P}\{d^{i}\} R_{\gamma}^{\beta}(v'; d^{i}) \right],$$

where: (a) follows from $\sum_{i} \mathbb{P}\{d^{i}\} = 1$; (b) holds since $v'$ and $d^{i}$ are feasible realizations; (c) follows from Fubini’s Theorem since one has $|R_{\gamma}^{\beta}(v; d)| \leq \frac{1}{T} |\pi^{H}(v; d)| + \frac{\gamma}{T} |\pi^{\beta}(v; d)| \leq \bar{v}(1 + \gamma)$ because
no strategy (even in hindsight) can achieve more that $T \bar{v}$; and (d) follows because any randomized strategy can be thought of as a probability distribution over deterministic algorithms.

Proof of Lemma A.2. To simplify notation we drop the dependence on $k$. We use the worst-case instance structure detailed in the proof of Theorem 1. Fix any deterministic bidding strategy $\beta \in \mathcal{B}$. Since $\beta$ is deterministic one has that $\pi^\beta(\mathbf{d}, \mathbf{v}) = \sum_{t=1}^{T} \mathbf{1}\{d_t \leq b_t^\beta\}(v_t - d_t)$ for any vectors $\mathbf{d}$ and $\mathbf{v}$ where $b_t^\beta$ is the bid dictated by $\beta$ at time $t$.

Let $\bar{v} = (\bar{v}, \ldots, \bar{v})$ be the valuation sequence and $\mathbf{d}^i \in \mathcal{D}$ be the sequence of competing bids. We denote by $b_t^i$ the bid in period $t$ under this input and $\beta$. We also denote the corresponding expenditure by $z_t^i := \mathbf{1}\{d_t^i \leq b_t^i\}d_t^i$, and the corresponding net utility by $u_t^i := \mathbf{1}\{d_t^i \leq b_t^i\}(\bar{v} - d_t^i)$. We further denote the history of decisions and observations under $\bar{v}$, $\mathbf{d}^i$ and $\beta$ by $H_t := \sigma\left(\{v_t^i, b_t^i, z_t^i, u_t^i\}_{\tau=1}^{t-1}, \bar{v}\right)$ for any $t \geq 2$, with $H_1 := \sigma(\bar{v})$.

We now define the sequence $\mathbf{x}$. For each $j \in \{1, \ldots, m\}$, define:

$$x_j := \sum_{t=(j-1)[\frac{T}{m}] + 1}^{j[\frac{T}{m}]} \mathbf{1}\{d_t^1 \leq b_t^1\},$$

where we denote by $b_t^1$ the bid at time $t$ under history $H_t$. Then, $x_j$ is the number of auctions won by $\beta$ throughout the $j$’th batch of $\lfloor T/m \rfloor$ auctions when the vector of best competitors’ bids is $\mathbf{d}^1$ and the vector of valuations is $\mathbf{v}$. Since $\mathbf{d}^1$, $\mathbf{v}$ and $\beta$ are deterministic, so is $\mathbf{x}$, and clearly $\mathbf{x} \in \mathcal{X}$ because $\beta$ is budget-feasible. Denote by $\beta^x$ a strategy that for each $j \in \{1, \ldots, m\}$ bids $d_j$ in the first $x_j$ auctions of batch $j$. When competing bids are $d_j$ in the $j$’th batch, this strategy is guaranteed to win $x_j$ auctions.

We next show that $\beta^x$ achieves the same performance as $\beta$. First, assume that the vector of competing bids is $\mathbf{d}^1$. Then, by construction:

$$\pi^\beta(\bar{v}, \mathbf{d}^1) = \sum_{t=1}^{T} \mathbf{1}\{d_t^1 \leq b_t^1\}(\bar{v} - d_t^1) = \sum_{j=1}^{m} x_j(\bar{v} - d_j) = \pi^{\beta^x}(\bar{v}, \mathbf{d}^1),$$

where $d_j$ denotes the competing bid of the $j$’th batch of the competing bid sequence $\mathbf{d}^1$. For any $\mathbf{d}^i \in \mathcal{D} \setminus \{\mathbf{d}^1\}$, define $\tau^i := \inf\{t \geq 1 : d_t^i \neq d_t^1\}$ to be the first time that the competing bid sequence is different than $\mathbf{d}^1$. Let $m^i \in \{1, \ldots, m\}$ be the number of batches that the competing bid sequence $\mathbf{d}^i$ has in common with $\mathbf{d}^1$. Since each batch has $\lfloor T/m \rfloor$ items, we have that $\tau^i = m^i \lfloor T/m \rfloor + 1$. The sequences $\mathbf{d}^1$ and $\mathbf{d}^i$ are identical, thus indistinguishable, up to time $\tau^i$. Therefore, the bids
of any deterministic strategy coincide up to time \( \tau^i \) under histories \( \mathcal{H}_t^1 \) and \( \mathcal{H}_t^i \). Then, one has:

\[
\pi^\beta(\tilde{v}, d^i) \equiv \sum_{t=1}^{\tau^i-1} \mathbf{1}\{d_t^i \leq b_t^i\}(\tilde{v} - d_t^i) = \sum_{t=1}^{\tau^i-1} \mathbf{1}\{d_t^1 \leq b_t^1\}(\tilde{v} - d_t^1) = \sum_{j=1}^{m_i} x_j(\tilde{v} - d_j) = \pi^{\beta^*}(\tilde{v}, d^i),
\]

where (a) follows because all items in periods \( t \in \{\tau^i, \ldots, T\} \) have zero utility under \( d^i \), (b) follows because the competing bid sequences and bids are equal during periods \( t \in \{1, \ldots, \tau^i - 1\} \), (c) follows from our definition of \( x \) and using that sequence \( d^i \) has \( m_i \) batches with nonzero utility. We have thus established that \( \pi^{\beta^*}(\tilde{v}, d^i) = \pi^{\beta^*}(\tilde{v}, d^i) \) for any \( d \in \mathcal{D} \). This concludes the proof. 

**Proof of Lemma A.3**. Let each advertiser \( k \) follow an adaptive pacing strategy, and define \( \delta_t = \sum_{k=1}^K E_k \left[ |\mu_{k,t} - \mu_k^i|^2 \right] \). Denote \( \Psi_k(\mu) = \Psi_k(\mu) / T \), and let \( \hat{\Psi}_{k,t}(\mu) = (v_{k,t} - (1 + \mu_{k,t})d_{k,t})^+ + \mu_{k,t} \rho_k \) be normalized empirical dual objective function for advertiser \( k \), with \( d_{k,t} = \max_{i \neq k} v_{i,t} / (1 + \mu_{i,t}) \).

By the structure of the strategy, the utility of advertiser \( k \) from the \( t \)th auction can be written as:

\[
u_{k,t} = 1\{d_{k,t}(1 + \mu_{k,t}) \leq v_{k,t}\}\{v_{k,t} - d_{k,t}\} = (v_{k,t} - (1 + \mu_{k,t})d_{k,t})^+ + \mu_{k,t} z_{k,t} = \hat{\Psi}_{k,t}(\mu_t) + \mu_{k,t}(z_{k,t} - \rho_k).
\]

Taking expectations we obtain:

\[E[\nu_{k,t}] \leq E\left[ \hat{\Psi}_{k,t}(\mu_t) + \mu_{k,t}(z_{k,t} - \rho_k) \mid \mu_t \right] \leq E[\hat{\Psi}_k(\mu_t)] + E[\mu_{k,t}(G_k(\mu_t) - \rho_k)] ,
\]

where (a) follows from the linearity of expectation and conditioning on the multipliers \( \mu_t \), and (b) holds since \( \{v_{k,t}\}_{k=1}^K \) are independent of the multipliers \( \mu_t \). Let \( \tilde{v} = \max_k \tilde{v}_k \). Since the dual objective is Lipschitz continuous (Lemma C.2 part (iii)) one has:

\[\Psi_k(\mu_t) \geq \Psi_k(\mu^*) - \tilde{v} \|\mu_t - \mu^*\|_1 ,
\]

for all \( t = 1, \ldots, T \) because \( a_{k,i} \in [0, 1] \). Since the expenditure is Lipschitz continuous (Lemma C.2 part (iii)) one has:

\[\mu_{k,t}(G_k(\mu_t) - \rho_k) = \mu_{k,t}(G_k(\mu_t) - G_k(\mu^*)) + (\mu_{k,t} - \mu_k^*)(G_k(\mu^*) - \rho_k) + \mu_k^*(G_k(\mu^*) - \rho_k) \geq -(\tilde{\mu}_k \tilde{v}^2 + \tilde{v}) \|\mu_t - \mu^*\|_1 ,
\]

for all \( t = 1, \ldots, T \), where (a) follows from \( \mu_{k,t} \leq \tilde{\mu}_k, |G_k(\mu^*) - \rho_k| \leq \tilde{v} \) together with \( \mu_k^*(G_k(\mu^*) - \rho_k) = \]
0, which follows the characterization of $\mu^*$ in (5). In addition, one has:

$$
\sum_{k=1}^{K} \mathbb{E}[\mu_{k,t} - \mu_k^*] \leq \mathbb{E} \left[ \left( K \sum_{k=1}^{K} |\mu_{k,t} - \mu_k^*|^2 \right)^{1/2} \right] \leq K^{1/2} \left( \sum_{k=1}^{K} \mathbb{E} [|\mu_{k,t} - \mu_k^*|^2] \right)^{1/2} = K^{1/2} \delta_t^{1/2}
$$

where (a) follows from $\sum_{i=1}^{n} |y_i| \leq (n \sum_{i=1}^{n} y_i^2)^{1/2}$, and (b) follows from Jensen’s inequality. Together, we obtain:

$$
\mathbb{E} [u_{k,t}] \geq \tilde{\Psi}_k(\mu^*) - (2\bar{v} + \mu_k \bar{v} f) K^{1/2} \delta_t^{1/2}.
$$

This concludes the proof. \qed

**Proof of Lemma A.4.** Denote by $\mu_t \in \mathbb{R}^K$ the (random) vector such that $\mu_{k,t} = \mu_k^*$ and $\mu_{i,t}$ is the multiplier used by advertiser $i \neq k$ at time $t$. Fix an advertiser $i \neq k$. Define $\delta_{i,t} := \mathbb{E} [(\mu_{i,t} - \mu_i^*)^2]$ and $\delta_t := \sum_{i \neq k} \delta_{i,t}$. Similarly, define $\hat{\delta}_{i,t} := \delta_{i,t}/\varepsilon_i$ and $\hat{\delta}_t := \sum_{i \neq k} \hat{\delta}_{i,t}$. We obtain from (A-20) in the proof of Theorem 4 that:

$$
\hat{\delta}_{i,t+1} \leq \hat{\delta}_{i,t} - 2\mathbb{E} [(\mu_{i,t} - \mu_i^*) (\rho_i - z_{i,t})] + \varepsilon_i \mathbb{E} [ |\rho_i - z_{i,t}|^2 ]
$$

$$
= \hat{\delta}_{i,t} - 2\mathbb{E} [(\mu_{i,t} - \mu_i^*) (\rho_i - \mathbb{E} [z_{i,t} | \mu_t])] + \varepsilon_i \mathbb{E} [ |\rho_i - z_{i,t}|^2 ] ,
$$

where the equality follows by conditioning on $\mu_t$. Let $\bar{v} = \max_{k \neq i} \bar{v}_i$. The third term satisfies $\mathbb{E} [ |\rho_i - z_{i,t}|^2 ] \leq \bar{v}^2$, since $\rho_i, z_{i,t} \in [0, \bar{v}]$. Proceeding to bound the second term, recall that the payment of advertiser $i$ is $z_{i,t} = 1 \{ d_{i,t} \leq v_{i,t}/(1 + \mu_{i,t}) \} d_{i,t}$ where the competing bid faced by the advertiser is given by

$$
d_{i,t} = \begin{cases} 
\beta_{k,t} \lor d_{i \backslash k,t} , & \text{if } m_{k,t} = m_{i,t} , \\
d_{i \backslash k,t} , & \text{if } m_{k,t} \neq m_{i,t} ,
\end{cases}
$$

where $d_{i \backslash k,t} = \max_{j \neq i, m_j = m_{i,t}} \{ v_{j,t}/(1 + \mu_{j,t}) \}$ denotes the maximum competing bid faced by advertiser $i$ when advertiser $k$ is excluded. Recall that for a fixed vector $\mu \in \mathbb{R}_+^K$, the expected expenditure function is given by $G_i(\mu) = \mathbb{E} \left[ \bar{d}_i 1 \left\{ \bar{d}_i \leq v_i/(1 + \mu_i) \right\} \right]$ where $\bar{d}_i = \max_{j \neq i, m_j = m_{i,t}} \{ v_{j,t}/(1 + \mu_j) \}$. For the second term in (B-1) one has:

$$
(\mu_{i,t} - \mu_i^*) (\rho_i - \mathbb{E} [z_{i,t} | \mu_t]) = (\mu_{i,t} - \mu_i^*) (\rho_i - G_i(\mu^*) + G_i(\mu^*) - G_i(\mu_t) + G_i(\mu_t) - \mathbb{E} [z_{i,t} | \mu_t])
$$

$$
\geq (\mu_{i,t} - \mu_i^*) (G_i(\mu^*) - G_i(\mu_t)) - |\mu_{i,t} - \mu_i^*| \cdot |G_i(\mu_t) - \mathbb{E} [z_{i,t} | \mu_t]| ,
$$

where the inequality follows because $\mu_{i,t} \geq 0$ and $\rho_i - G_i(\mu^*) \geq 0$ and $\mu_i^* (\rho_i - G_i(\mu^*)) = 0$, together
with $xy \geq -|x| \cdot |y|$ for $x, y \in \mathbb{R}$. Because values are independent, and advertisers $i$ and $k$ compete only when $m_{k,t} = m_{i,t}$, we obtain that

$$|G_i(\mu_t) - \mathbb{E}[z_{i,t} | \mu_t]| = \left| \mathbb{E} \left[ \left( \tilde{d}_{i,t} 1\{\tilde{d}_{i,t} \leq b_{i,t}\} - d_{i,t} 1\{d_{i,t} \leq b_{i,t}\} \right) 1\{m_{k,t} = m_{i,t}\} \right] \right| \leq \tilde{v} a_{k,i},$$

where the equality follows because the bid of advertiser $i \neq k$ is $b_{i,t} = v_{i,t}/(1 + \mu_{i,t})$, and $d_{i,t} = b_{k,t}^\beta \vee d_{i\setminus k,t}$ and $\tilde{d}_{i,t} = (v_{k,t}/(1 + \mu_k^*)) \vee d_{i\setminus k,t}$ when $m_{k,t} = m_{i,t}$; and the inequality follows because the expenditure is at most the bid and $b_{i,t} \leq \tilde{v}$ together with $\mathbb{P}\{m_{k,t} = m_{i,t}\} = a_{k,i}$. Summing over the different advertisers and using Assumption 2 part (1) we obtain by

$$
\sum_{i \neq k} (\mu_{i,t} - \mu_k^*) \left( \rho_i - \mathbb{E}[z_{i,t} | \mu_t] \right) \geq \sum_{i=1}^{K} \frac{1}{2} \left( \mu_{i,t} - \mu_k^* \right)^2 (G_i(\mu^*) - G_i(\mu_t)) - \tilde{v} \sum_{i \neq k} a_{k,i} |\mu_{i,t} - \mu_k^*| \\
\geq \lambda \|\mu_t - \mu^*\|_2^2 - \tilde{v} \|a_k\|_2 \cdot \|\mu_t - \mu^*\|_2^2,
$$

where the last inequality follows from Cauchy-Schwarz inequality. Denoting $\tilde{\epsilon} = \max_{k \in \{1, \ldots, K\}} \epsilon_k$, $\xi = \min_{k \in \{1, \ldots, K\}} \epsilon_k$, we conclude by summing (A-20) over $k$ that:

$$
\hat{\delta}_{t+1} \overset{(a)}{\leq} \hat{\delta}_t - 2\lambda \delta_t + \epsilon K \tilde{v}^2 + 2\tilde{v} \|a_k\|_2 \delta_t^{1/2} \overset{(b)}{\leq} (1 - 2\lambda \xi) \hat{\delta}_t + \epsilon K \tilde{v}^2 + 2\tilde{v} \|a_k\|_2 \tilde{v} \tilde{\epsilon}^{1/2} \tilde{\delta}_t^{1/2},
$$

where (a) follows because $\mathbb{E}\|\mu_t - \mu^*\|_2 \leq \left( \mathbb{E}\|\mu_t - \mu^*\|_2^2 \right)^{1/2} = \delta_t^{1/2}$ by Jensen’s Inequality, and (b) follows from $\delta_t = \sum_{i \neq k} \hat{\delta}_{i,t} = \sum_{i \neq k} \epsilon^\ast \hat{\delta}_{i,t} \geq \epsilon \tilde{\delta}_t$ because $\hat{\delta}_{i,t} \geq 0$ and $\delta_t \leq \epsilon \tilde{\delta}_t$. Lemma C.5 with $a = 2\lambda \xi \leq 1$, $b = \epsilon K \tilde{v}^2$ and $c = 2\tilde{v} \|a_k\|_2 \epsilon^{1/2}$ implies that

$$
\hat{\delta}_t \leq \hat{\delta}_1 (1 - \lambda \xi)^{t-1} + \frac{K \tilde{v}^2 \epsilon}{\lambda \xi} \left( \frac{\tilde{v} \|a_k\|_2 \epsilon^{1/2}}{\lambda \xi} \right)^2.
$$

Using that $\delta_t \leq \epsilon \tilde{\delta}_t$ together with $\tilde{\delta}_1 \leq \delta_1 / \xi \leq K \tilde{\mu}^2 / \epsilon$ because $\mu_{i,t}, \mu_k^* \in [0, \tilde{\mu}]$ and $\tilde{\mu} = \max_k \tilde{\mu}_k$ we obtain that

$$
\delta_t \leq K \tilde{\mu}^2 \epsilon \xi (1 - \lambda \xi)^{t-1} + \frac{K \tilde{v}^2 \epsilon^2}{\lambda \xi} + \frac{\tilde{v}^2 \|a_k\|_2^2 \epsilon^2}{\xi^2}.
$$

This concludes the proof.

**Proof of Lemma A.5.** Let each advertiser $i \neq k$ follow the adaptive pacing strategy. Denote $\Psi_k(\mu) := \Psi_k(\mu)/T$, and by $\mu_t \in \mathbb{R}^K$ the (random) vector such that $\mu_{k,t} = \mu_k^*$ and $\mu_{i,t}$ is the multiplier of the adaptive pacing strategy of advertiser $i \neq k$ at time $t$. Based on the second-price
allocation rule, the Lagrangian utility of advertiser \( k \) from the \( t \)th auction can be written as:

\[
\begin{align*}
    u_{k,t} - \mu_k^* z_{k,t} + \rho_k &= 1\{d_{k,t} \leq b_{k,t}^\beta\} (v_{k,t} - (1 + \mu_k^*) d_{k,t}) + \rho_k \\
    &\leq (v_{k,t} - (1 + \mu_k^*) d_{k,t})^+ + \rho_k,
\end{align*}
\]

where the first equality follows because \( z_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} d_{k,t} \) and \( u_{k,t} = 1\{d_{k,t} \leq b_{k,t}^\beta\} (v_{k,t} - d_{k,t}) \), and the inequality because \( x \leq x^+ \) for all \( x \in \mathbb{R} \) and dropping the indicator that advertiser \( k \) wins the auction. Taking expectations we obtain:

\[
\mathbb{E}[u_{k,t} - \mu_k^* z_{k,t} + \rho_k] \leq \mathbb{E}\left[\mathbb{E}\left[(v_{k,t} - (1 + \mu_k^*) d_{k,t})^+ | \mu_t\right] + \rho_k\right] \overset{(b)}{=} \mathbb{E}\left[\bar{\Psi}_k(\mu_t)\right],
\]

where \((a)\) follows from conditioning on \( \mu_t \), and \((b)\) holds since \( \{v_{k,t}\}_{k=1}^K \) are independent of the multipliers \( \mu_t \). Since the dual objective is Lipschitz continuous (Lemma C.2, part (ii)) one has:

\[
\bar{\Psi}_k(\mu_t) \leq \bar{\Psi}_k(\mu^*) + \bar{v}_k \sum_{i \neq k} a_{k,i} |\mu_{i,t} - \mu_i^*| \leq \bar{\Psi}_k(\mu^*) + \bar{v}_k \|a_k\|_2 \cdot \|\mu_t - \mu^*\|_2,
\]

where the last inequality follows from Cauchy-Schwarz inequality, for all \( t = 1, \ldots, T \). The result follows by taking expectations and using Jensen’s Inequality.

\[\square\]

C Additional auxiliary analysis

The following result provides sufficient conditions for the dual function to be thrice differentiable and strongly convex. These conditions are required by Theorem 3 to show the asymptotic optimality of an adaptive pacing strategy in stationary settings. Here we assume that values and competing bids \((v_k, d_k)\) are independently drawn from some stationary distribution. Recall that the dual function is \( \Psi_k(\mu) := T \mathbb{E}_{v_k, d_k} [(v_k - (1 + \mu) d_k)^+] + \mu \rho_k \) and the expenditure function is \( G_k(\mu) := \mathbb{E}_{v_k, d_k} [1\{(1 + \mu) d_k \leq v_k\} d_k] \).

**Lemma C.1.** Suppose that (a) the valuation \( v_k \) has a density satisfying \( 0 < f \leq f_k(x) \leq \bar{f} < \infty \) for all \( x \in [0, \bar{v}_k] \), (b) the valuation density is differentiable with bounded derivative \( |f_k'(x)| \leq \bar{f}' \) for all \( x \in [0, \bar{v}_k] \), (c) the competing bid \( d_k \) is independent of \( v_k \) and absolute continuous with density \( h_k(x) \) satisfying \( 0 < \bar{h} \leq h_k(x) \leq \bar{h} < \infty \) for all \( x \in [0, \bar{v}_k] \). Then:

(i) \( \Psi_k(\mu) \) is differentiable with derivative \( \Psi_k'(\mu) = T(\rho_k - G_k(\mu)) \).
(ii) $G_k(\mu)$ is differentiable and there exist $G'_k > 0$ and $\lambda > 0$ such that the derivative is bounded by $-G'_k \leq G'_k(\mu) \leq -\lambda_k < 0$ for all $\mu \in [0, \bar{\mu}_k]$. This implies that $\Psi_k(\mu)$ is strongly convex with parameter $\lambda_k$.

(iii) $G_k(\mu)$ is twice-differentiable and there exists $G''_k > 0$ such that the second derivative is upper bounded by $|G''_k(\mu)| \leq G''_k$ for all $\mu \in [0, \bar{\mu}_k]$.

**Proof of Lemma C.1.** We drop the dependence on $k$ to simplify the notation. Part (i) follows from Lemma C.2, part (ii). We prove the other parts of the Lemma.

(ii). From Lemma C.2, part (iii) we know that $G$ is differentiable with derivative given in (B-2). Moreover, we have that $G'(\mu) \geq -\bar{v}^2 \bar{f}$. We can upper bound the derivative as follows:

$$G'(\mu) = -\int_0^{\bar{v}/(1+\mu)} x^2 f((1 + \mu)x) \, dH(x) \leq -\bar{f} h \int_0^{\bar{v}/(1+\mu)} x^2 \, dx = -\frac{\bar{f} h \bar{v}^3}{3(1+\mu)^3},$$

where the first inequality follows because the densities are lower bounded, and using that $\mu \leq \bar{\mu}$ together with the fact that the integrand is positive.

(iii). Using (B-2) we obtain that the second derivative of the expenditure function is

$$G''(\mu) = \frac{\bar{v}^3}{(1+\mu)^2} f(\bar{v})(1+\mu) - \int_0^{\bar{v}/(1+\mu)} x^3 f'((1 + \mu)x) \, dH(x),$$

where we used Leibniz rule because $x^2 f((1 + \mu)x)$ is differentiable almost everywhere w.r.t. $\mu$ with derivative that is bounded by $\bar{v}^3 f'$. A similar argument as in part (iii) yields that

$$|G''(\mu)| \leq \bar{v}^3 \bar{f} h + \frac{\bar{f}' h \bar{v}^4}{4}. \tag*{□}$$

The following result obtains some key characteristics of the model primitives under the matching model described in Section 5. We denote by $a_{k,i} = \mathbb{P}\{m_{k,t} = m_{i,t}\} = \sum_{m=1}^{M} \alpha_{k,m} \alpha_{i,m}$ the probability that advertisers $k$ and $i$ compete in the same auction at a given time period. Given a vector of multipliers $\mu$, we denote $\Psi_k(\mu) := T(\mathbb{E}_{v,m}[(v_k - (1 + \mu_k)d_k)^+] + \mu_k \rho_k)$ the dual performance under $\mu$ with $d_k = \max\left\{ \max_{i: i \neq k} \{1\{m_i = m_k\}v_i/(1 + \mu_i)\}, 0 \right\}$ and the expectation taken w.r.t. $v_i$ and $m_i$ for all $i$. In addition, we denote by $G_k(\mu) := \mathbb{E}_{v,m} [1\{(1 + \mu_k)d_k \leq v_k\}] d_k$ the expected expenditure under the second-price auction allocation rule. Results for the original model hold by putting $M = 1$ and $\alpha_{k,m} = 1$. 

27
Lemma C.2. Suppose that for each advertiser \( k \) the valuation density satisfies \( f_k(x) \leq \bar{f} < \infty \) for all \( x \in [0, \bar{v}_k] \). Then:

(i) Fix \( \mu_{k,t} = (\mu_{i,t})_{i \neq k} \in \mathbb{R}^{K-1} \), the competitor’s multipliers at time \( t \). The maximum competing bid \( d_{k,t} = \max_{i \neq k, m_{k,t} = m_{i,t}} \{ v_{i,t}/(1 + \mu_{i,t}) \} \) is integrable over \([0, \bar{d}_{k,t}]\) where \( \bar{d}_{k,t} = \max_{i \neq k} \{ \bar{v}_i/(1 + \mu_{i,t}) \} \), with cumulative distribution function

\[
H_k(x; \mu_{k,t}) = \prod_{i \neq k} \left( 1 - a_{k,i} \bar{F}_i((1 + \mu_{i,t})x) \right).
\]

(ii) \( \Psi_k(\cdot) \) is differentiable and Lipschitz continuous. In particular, for any \( \mu \in \mathcal{U} \) and \( \mu' \in \mathcal{U} \), one has

\[
\frac{1}{T} |\Psi_k(\mu) - \Psi_k(\mu')| \leq \bar{v}_k |\mu_k - \mu'_k| + \bar{v}_k \sum_{i \neq k} a_{k,i} |\mu_i - \mu'_i|.
\]

(iii) \( G_k(\cdot) \) is Lipschitz continuous. In particular, for any \( \mu \in \mathcal{U} \) and \( \mu' \in \mathcal{U} \), one has

\[
|G_k(\mu) - G_k(\mu')| \leq \bar{v}_k^2 \bar{f}_k |\mu_k - \mu'_k| + 2\bar{v}_k \bar{f}_k \sum_{i \neq k} a_{k,i} |\mu_i - \mu'_i|.
\]

**Proof of Lemma C.2.** We prove the three parts of the Lemma.

(i). Using that the values \( v_{i,t} \) are independent across advertisers and identical across time, we can write the cumulative distribution function as

\[
H_k(x; \mu_{k,t}) = \mathbb{P}\left\{ \max_{i \neq k, m_{k,t} = m_{i,t}} \frac{v_{i,t}}{1 + \mu_{i,t}} \leq x \right\} = (a) \prod_{i \neq k} \left( \mathbb{P}\left\{ m_{k,t} \neq m_{i,t} \right\} + \mathbb{P}\left\{ m_{k,t} = m_{i,t} \right\} \mathbb{P}\left\{ \frac{v_i}{1 + \mu_{i,t}} \leq x \right\} \right) = (b) \prod_{i \neq k} \left( 1 - a_{k,i} \bar{F}_i((1 + \mu_{i,t})x) \right),
\]

where \((a)\) follows by conditioning on whether advertiser \( i \neq k \) participates in the same auction that advertiser \( k \), and \((b)\) follows from \( a_{k,i} = \mathbb{P}\{ m_{k,t} = m_{i,t} \} \). Because \( F_i(\cdot) \) has support \([0, \bar{v}_i]\) we conclude that the support is \([0, \bar{d}_{k,t}]\) with \( \bar{d}_{k,t} = \max_{i \neq k} \bar{v}_i/(1 + \mu_{i,t}) \).

(ii). Denote by \( \bar{\Psi}_k(\mu) = \Psi_k(\mu)/T \). For every realized vectors \( v = \{v_i\}_i \) and \( m = \{m_i\}_i \), the function \( (v_k - (1 + \mu_k)d_k)^+ \) is differentiable in \( \mu_k \) with derivative \(-d_k1\{v_k \geq (1 + \mu_k)d_k\}\), except in the set \( \{(v,m) : v_k = (1 + \mu_k)d_k\} \) that has measure zero because values are absolutely continuous.
with support $[0, \bar{v}_k]$ and independent. As the derivative is bounded by $d_k$, which is integrable since $d_k \leq \bar{v}$ with $\bar{v} := \max_k \bar{v}_k$ from Item (i), we conclude by Leibniz’s integral rule that:

$$\frac{\partial \Psi_k(\mu)}{\partial \mu_k} = \rho_k - E[d_k 1\{v_k \geq (1 + \mu_k)d_k\}] = \rho_k - G_k(\mu),$$

which implies that $|\frac{\partial \Psi_k(\mu)}{\partial \mu_k}| \leq \bar{v}_k$ because $\rho_k, G_k(\mu) \in [0, \bar{v}_k]$.

Fix an advertiser $i \neq k$. Recall that the maximum competing bid faced by advertiser $k$ is $d_k = \max_{i \neq k, m_k = m_i} \{v_i/(1 + \mu_i)\}$. Let $d_{k\setminus i} = \max_{j \neq k, i; m_k = m_j} \{v_j/(1 + \mu_j)\}$ be the maximum competing bid faced by advertiser $k$ with advertiser $i$ excluded. By conditioning on whether advertiser $i$ and $k$ compete in the same auction, we can write the random function $(v_k - (1 + \mu_k)d_k)^+$ as

$$(v_k - (1 + \mu_k)d_k)^+ = \begin{cases} (v_k - (1 + \mu_k)d_{k\setminus i} \lor \frac{v_i}{1 + \mu_i})^+, & \text{if } m_{k,t} = m_{i,t}, \\
(v_k - (1 + \mu_k)d_{k\setminus i})^+, & \text{if } m_{k,t} \neq m_{i,t}. \end{cases}$$

We obtain that the function $(v_k - (1 + \mu_k)d_k)^+$ is differentiable in $\mu_i$, with derivative

$$v_i \frac{1 + \mu_k}{(1 + \mu_i)^2} \left\{ \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} \geq d_{k\setminus i}, m_k = m_i \right\}$$

except in the sets $\{(v, m) : \frac{v_k}{1 + \mu_k} = \frac{v_i}{1 + \mu_i} \geq d_{k\setminus i}, m_k = m_i\}$ and $\{(v, m) : \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} = d_{k\setminus i}, m_k = m_i\}$. Again, these sets have measure zero because values are absolutely continuous with support $[0, \bar{v}]$ and independent. Because the derivative is bounded by $v_k/(1 + \mu_i)$, which is integrable since $v_k \leq \bar{v}$, we conclude by Leibniz’s integral rule that:

$$\frac{\partial \Psi_k(\mu)}{\partial \mu_i} = E \left[v_i \frac{1 + \mu_k}{(1 + \mu_i)^2} 1 \left\{ \frac{v_k}{1 + \mu_k} \geq \frac{v_i}{1 + \mu_i} \geq d_{k\setminus i}, m_k = m_i \right\} \right]$$

$$\leq E \left[\frac{v_k}{1 + \mu_i} 1\{m_k = m_i\}\right] \leq \bar{v}_k a_{k,i},$$

where the first inequality follows because $\frac{v_i}{1 + \mu_i} \leq \frac{v_k}{1 + \mu_k}$ and dropping part of the indicator, and the last inequality follows because $v_k \in [0, \bar{v}_k]$ and $\mu_i \geq 0$. This concludes the proof.

(iii). We show that $G_k(\mu)$ is Lipschitz continuous by bounding its derivatives. Since values are independent across advertisers we can write the expected expenditure as
\[ G_k(\mu) = \int_0^{\bar{v}_k} x \bar{F}_k((1 + \mu_k)x) \, dH_k(x; \mu_k) \]
\[ = \int_0^{\bar{v}_k} ((1 + \mu_k)x f_k((1 + \mu_k)x) - \bar{F}_k((1 + \mu_k)x)) \, H_k(x; \mu_k) \, dx, \]

where the second equation follows from integration by parts. Using the first expression for the expected expenditure we obtain that
\[
\frac{\partial G_k}{\partial \mu_k}(\mu) = -\int_0^{\bar{v}_k} x^2 f_k((1 + \mu_k)x) 1 \{ (1 + \mu_k)x \leq \bar{v}_k \} \, dH_k(x; \mu_k),
\]

where we used Leibniz rule because \( x \bar{F}_k((1 + \mu_k)x) \) is differentiable w.r.t. \( \mu_k \) almost everywhere with derivative that is bounded by \( \bar{v}_k^2 \bar{f} \). Therefore, one obtains
\[
\left| \frac{\partial G_k}{\partial \mu_k}(\mu) \right| \leq \bar{v}_k^2 \bar{f}.
\]

Using the second expression for the expected expenditure we obtain for \( i \neq k \) that
\[
\frac{\partial G_k}{\partial \mu_i}(\mu) = \int_0^{\bar{v}_k} ((1 + \mu_k)x f_k((1 + \mu_k)x) - \bar{F}_k((1 + \mu_k)x)) \, \frac{\partial H_k}{\partial \mu_i}(x; \mu_k) \, dx,
\]

where we used Leibniz rule as \( H_k(x; \mu_k) \) is differentiable w.r.t. \( \mu_i \) almost everywhere with derivative
\[
\frac{\partial H_k}{\partial \mu_i}(x; \mu_k) = a_{k,i} x f_i((1 + \mu_i)x) 1 \{ (1 + \mu_i)x \leq \bar{v}_i \} \prod_{j \neq k,i} (1 - a_{k,j} \bar{F}_j((1 + \mu_j)x)),
\]

which is bounded from above by \( a_{k,i} \bar{v}_k \bar{f} 1 \{ (1 + \mu_i)x \leq \bar{v}_i \} \). Therefore, one obtains
\[
\left| \frac{\partial G_k}{\partial \mu_i}(\mu) \right| \leq 2 a_{k,i} \bar{v}_k^2 \bar{f},
\]

and the result follows. □

### C.1 Auxiliary results

**Proposition C.3. (Uniqueness of steady state)** Suppose Assumption 2 holds. Then there exists a unique vector of multipliers \( \mu^* \in \mathcal{U} \) defined by (5).

**Proof.** We first establish that selecting a multiplier outside of \([0, \mu_k]\) is a dominated strategy for advertiser \( k \). Notice that for every \( \mu_{-k} \) and \( x > \tilde{\mu}_k \) we have that
\[
\Psi_k(x, \mu_{-k}) = \mathbb{E}_v \left[ (v_k - (1 + x)d_k)^+ \right] + x\rho_k \geq x\rho_k \geq \bar{v} \geq \Psi_k(0, \mu_{-k}),
\]

where (a) follows from dropping the first term, (b) holds since by Assumption 2 one has that \(\rho_k \geq \bar{v}/\bar{\mu}_k\) from and \(x > \bar{\mu}_k\), and (c) follows from \(0 \leq v_k \leq \bar{v}\). Thus every \(x > \bar{\mu}_k\) in the dual problem is dominated by \(x = 0\), and the equilibrium multipliers lie in the set \(U\). Define \(G_k(\mu) := \mathbb{E}_v [1 \{ (1 + \mu_k)d_k \leq v_k \} d_k]\) to be the expected expenditure under the second-price auction allocation rule. Assumption 2 implies that:

\[
(\mu - \mu^*)^T (G(\mu^*) - G(\mu)) > 0,
\]

for all \(\mu \in U\) such that \(\mu \neq \mu^*\). To prove uniqueness, suppose that there exists another equilibrium multiplier \(\mu \in U\) such that \(\mu \neq \mu^*\). From (B-4) one has:

\[
0 < \sum_{k=1}^{K} (\mu_k - \mu^*_k)(G_k(\mu^*) - \rho_k + \rho_k - G_k(\mu)) = \sum_{k=1}^{K} \mu_k(G_k(\mu^*) - \rho_k) - \mu^*_k(\rho_k - G_k(\mu)),
\]

where (a) follows from \(\mu_k(\rho_k - G_k(\mu)) = 0\) and \(\mu^*_k(G_k(\mu^*) - \rho_k) = 0\) by (5). As \(\mu_k, \mu^*_k \geq 0\) and \(G_k(\mu^*), G_k(\mu) \leq \rho_k\), we obtain that the right hand-side is non-positive, contradicting (B-4).

**Lemma C.4.** Let \(\{\delta_t\}_{t \geq 1}\) be a sequence of numbers such that \(\delta_t \geq 0\) and \(\delta_{t+1} \leq (1-a)\delta_t + b_t\) with \(b_t \geq 0\) and \(0 \leq a \leq 1\). Then,

\[
\delta_t \leq (1-a)^{t-1}\delta_1 + \sum_{s=1}^{t-1} (1-a)^{t-1-s}b_s.
\]

When \(b_t = b\) for all \(t \geq 1\) and \(a > 0\), we obtain

\[
\delta_t \leq (1-a)^{t-1}\delta_1 + \frac{b}{a}.
\]

**Proof.** We prove the result by induction. The result trivially holds for \(t = 1\) because \(a, b_1 \geq 0\). For \(t > 1\), the recursion gives

\[
\delta_{t+1} \leq (1-a) \delta_t + b_t \leq (1-a)^t \delta_1 + \sum_{s=1}^{t} (1-a)^{t-s}b_s,
\]

31
where the second inequality follows from the induction hypothesis and the fact that \(1 - a \geq 0\). The last inequality follows because \(\sum_{s=1}^{t-1}(1-a)^t-1-s \leq \sum_{s=0}^{\infty}(1-a)^s = 1/a\) when \(a \in (0,1]\).

\[\text{Lemma C.5. Let } \{\delta_t\}_{t \geq 1} \text{ be a sequence of numbers such that } \delta_t \geq 0 \text{ and } \delta_{t+1} \leq (1-a)\delta_t + b + c\delta_t^{1/2} \text{ with } c \geq 0, b \geq 0 \text{ and } 0 \leq a \leq 1. \text{ Then,} \]

\[\delta_t \leq (1-a/2)^{t-1}\delta_1 + \frac{2b}{a} + \frac{c^2}{a^2}.\]

Proof. The square root term can be bounded as follows

\[c\delta_t^{1/2} = \frac{c}{a^{1/2}}a^{1/2}\delta_t^{1/2} \leq \frac{c^2}{2a} + \frac{a\delta_t}{2},\]

because \(xy \leq (x^2 + y^2)/2\) for \(x, y \in \mathbb{R}\) by the AM-GM inequality. Using this bound, we can rewrite the inequality in the statement as

\[\delta_{t+1} \leq (1-a)\delta_t + b + c\delta_t^{1/2} \leq (1-a/2)\delta_t + b + c^2/2a.\]

The result then follows from Lemma C.4.

\[\text{□}\]

C.2 Stability analysis

We first show that the first part of Assumption 2 can be implied by the diagonal strict concavity condition defined in Rosen (1965). Indeed, since the set \(U\) is compact and since the vector function \(G(\cdot)\) is bounded in \(U\), to verify that the first part of Assumption 2 holds, it suffices to show that

\[(\mu - \mu')^T(G(\mu') - G(\mu)) > 0 \text{ for all } \mu, \mu' \in U.\]

The latter is equivalent to the diagonal strict concavity assumption of Rosen (1965). Furthermore, denote by \(J_G : \mathbb{R}_+^K \rightarrow \mathbb{R}^{K \times K}\) the Jacobian matrix of the vector function \(G\), that is, \(J_G(\mu) = \left(\frac{\partial G_k}{\partial \mu_i}(\mu)\right)_{k,i}\). Then, by Theorem 6 of Rosen (1965), it is sufficient to show that the symmetric matrix \(J_G(\mu) + J_G^T(\mu)\) is negative definite.

We next provide an analytical expressions for \(G(\mu)\) for two advertisers with valuations that are independently uniformly distributed and exponentially distributed to demonstrate numerically that the latter condition holds in these cases.

Example C.6. (Two bidders with uniform valuations) Assume \(K = 2, U = [0,2]^2\), and \(v_{k,t} \sim U[0,1], \text{ i.i.d. for all } k \in \{1,2\} \text{ and } t \in \{1,\ldots,T\}. \text{ One obtains:} \)
\[ G_1(\mu) = \int_0^1 \int_0^{\min\left\{ \frac{1+\mu_2}{1+\mu_1}x_1, 1 \right\}} \frac{x_2}{1+\mu_2} dx_2 dx_1 \]
\[ = 1\{\mu_2 \leq \mu_1\} \frac{1 + \mu_2}{6(1+\mu_1)^2} + 1\{\mu_1 < \mu_2\} \left( \frac{1}{2(1+\mu_2)} - \frac{1 + \mu_1}{3(1+\mu_2)^2} \right). \]

Therefore:
\[ G(\mu) = 1\{\mu_2 \leq \mu_1\} \left[ \frac{1 + \mu_2}{6(1+\mu_1)^2} \right] + 1\{\mu_1 < \mu_2\} \left[ \frac{1}{2(1+\mu_2)} - \frac{1 + \mu_1}{3(1+\mu_2)^2} \right]. \]

Following this expression, one may validate the first part of Assumption 2 by creating a grid of \( \mu_{i,j} \in \mathcal{U} \), and for a given grid calculate the maximal monotonicity constant \( \lambda \) for which the condition holds. For example, for a 20 \( \times \) 20 grid (with \( \|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\| = 0.1 \) for all \( i = 0,1,\ldots,19 \) and \( j = 0,1,\ldots,19 \) the latter condition holds with \( \lambda = 0.013 \). Similarly, for a 40 \( \times \) 40 grid (with \( \|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\| = 0.05 \) for all \( i = 0,1,\ldots,39 \) and \( j = 0,1,\ldots,39 \) the latter condition holds with \( \lambda = 0.0127 \).

Example C.7. (Two bidders with exponential valuations) Assume \( K = 2 \), \( \mathcal{U} = [0,2]^2 \), and \( v_{k,t} \sim \exp(1) \), i.i.d. for all \( k \in \{1,2\} \) and \( t \in \{1,\ldots,T\} \). One obtains:
\[ G_1(\mu) = \int_0^\infty \int_0^{\min\left\{ \frac{1+\mu_2}{1+\mu_1}x_1, 1 \right\}} \frac{x_2 e^{-x_2}}{1+\mu_2} dx_2 dx_1 = \frac{1 + \mu_2}{(2 + \mu_2 + \mu_1)^2}. \]

Following same lines as in Example C.7, one may validate the first part of Assumption 2. For example, for a 20 \( \times \) 20 grid (with \( \|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\| = 0.1 \) for all \( i = 0,1,\ldots,19 \) and \( j = 0,1,\ldots,19 \) the latter condition holds with \( \lambda = 0.0295 \). Similarly, for a 40 \( \times \) 40 grid (with \( \|\mu_{i,j} - \mu_{i,j+1}\| = \|\mu_{i,j} - \mu_{i+1,j}\| = 0.05 \) for all \( i = 0,1,\ldots,39 \) and \( j = 0,1,\ldots,39 \) the latter condition holds with \( \lambda = 0.0286 \).

The following result expands and complements the above examples by showing that when the number of players is large, the stability assumption holds in symmetric settings in which every advertiser participates in each auction with the same probability and all advertisers have the same distribution of values. In particular, the monotonicity constant \( \lambda \) of the expenditure function \( G \) is shown to be independent of the number of players.

**Proposition C.8. (Stability in symmetric settings)** Consider a symmetric setting in which advertisers participate in each auction with probability \( \alpha_{i,m} = 1/M \), advertisers values are drawn...
from a continuous density \( f(\cdot) \), and the ratio of number of auctions to number of players \( \kappa := K/M \) is fixed. Then, there exist \( K \in \mathbb{N} \) and \( \lambda > 0 \) such that for all \( K \geq K \) there exists a set \( U \subset \mathbb{R}_+^K \) with \( 0 \in U \) such that \( G \) is \( \lambda \)-strongly monotone over \( U \).

Proof. We prove the result in three steps. First, we argue that a sufficient condition for \( \lambda \)-strong monotonicity of \( G \) over \( U \) is that \( \lambda \) is a lower bound on the minimum eigenvalue of the symmetric part of the Jacobian of \( -G \) over \( U \). Second, we characterize the Jacobian of the vector function \( G \) in the general case. Third, we show that in the symmetric case \( G \) is strongly monotone around \( \mu = 0 \) by bounding the minimum eigenvalue of the symmetric part of the Jacobian of \( -G \) at zero.

Step 1. We denote by \( J_G : \mathbb{R}_+^K \rightarrow \mathbb{R}_+^{K \times K} \) the Jacobian matrix of the vector function \( G \), that is, \( J_G(\mu) = \left( \frac{\partial G_k}{\partial \mu_i}(\mu) \right)_{k,i} \). Let \( \lambda \) be a lower bound on the minimum eigenvalue of the symmetric part of \( -J_G(\mu) \) over \( U \). That is, \( \lambda \) satisfies

\[
\lambda \leq \min_{\|x\|_2 = 1} \frac{1}{2} x^T (J_G(\mu) + J_G(\mu)^T) x
\]

for all \( \mu \in U \). Lemma C.2 part (iii) shows that \( G(\mu) \) is differentiable in \( \mu \). Thus, by the mean value theorem there exists some \( \xi \in \mathbb{R}_+^K \) in the segment between \( \mu \) and \( \mu' \) such that for all \( \mu, \mu' \in U \) one has \( G(\mu') = G(\mu) + J_G(\xi)(\mu' - \mu) \). Therefore,

\[
(\mu - \mu')^T (G(\mu') - G(\mu)) = -\frac{1}{2}\left( \mu - \mu' \right)^T (J_G(\xi) + J_G(\xi)^T) (\mu - \mu') \geq \lambda \| \mu - \mu' \|^2_2,
\]

since \( \xi \in U \). Hence, a sufficient condition for \( \lambda \)-strong monotonicity of \( G \) over \( U \) is that \( \lambda \) is a lower bound on the minimum eigenvalue of the symmetric part of the Jacobian of \( -G \) over \( U \).

Step 2. Some definitions are in order. Let \( H_{k,i} (x; \mu_{-k,i}) = \prod_{j \neq i,k} \left( 1 - a_{k,j} \bar{F}_i((1 + \mu_j)x) \right) \) and \( \ell_k(x) = x f_k(x) 1 \{ x \leq \bar{v}_k \} \). Additionally, we denote

\[
\gamma_{k,i} = \int_0^{\bar{v}_k} \ell_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i} (x; \mu_{-k,i}) \, dx,
\]

\[
\omega_{k,i} = \int_0^{\bar{v}_k} \bar{F}_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i} (x; \mu_{-k,i}) \, dx.
\]

We first determine the partial derivatives of the cumulative distribution function \( H_k \). One has
\[
\frac{\partial H_k}{\partial x}(x; \mu_k) = \sum_{i \neq k} a_{k,i}(1 + \mu_i)f_i ((1 + \mu_i)x) \mathbf{1}\{(1 + \mu_i)x \leq \bar{v}\} H_{k,i}(x; \mu_{-k,i}),
\]

\[
\frac{\partial H_k}{\partial \mu_i}(x; \mu_k) = a_{k,i}xf_i ((1 + \mu_i)x) \mathbf{1}\{(1 + \mu_i)x \leq \bar{v}\} H_{k,i}(x; \mu_{-k,i}).
\]

Using equation (B-2) one obtains that

\[
\frac{\partial G_k}{\partial \mu_k} (\mu) = - \int_0^{\bar{v}_k} x^2 f_k ((1 + \mu_k)x) \mathbf{1}\{(1 + \mu_k)x \leq \bar{v}\} dH_k(x; \mu_k)
\]

\[
= - \sum_{i \neq k} \frac{a_{k,i}}{1 + \mu_k} \int_0^{\bar{v}_k} \ell_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{-k,i}) dx
\]

\[
= - \sum_{i \neq k} \frac{a_{k,i}}{1 + \mu_k} \gamma_{k,i}.
\]

Using equation (B-3) we have

\[
\frac{\partial G_k}{\partial \mu_i} (\mu) = \int_0^{\bar{v}_k} ((1 + \mu_k)x f_k ((1 + \mu_k)x) - \bar{F}_k ((1 + \mu_k)x)) \frac{\partial H_k}{\partial \mu_k}(x; \mu_k) dx
\]

\[
= \frac{a_{k,i}}{1 + \mu_i} \int_0^{\bar{v}_k} \ell_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{-k,i}) dx
\]

\[
- \frac{a_{k,i}}{1 + \mu_i} \int_0^{\bar{v}_k} \bar{F}_k ((1 + \mu_k)x) \ell_i ((1 + \mu_i)x) H_{k,i}(x; \mu_{-k,i}) dx
\]

\[
= \frac{a_{k,i}}{1 + \mu_i} (\gamma_{k,i} - \omega_{k,i}).
\]

**Step 3.** Consider a symmetric setting in which each advertiser participates in each auction with the same probability and all advertisers have the same distribution of values. By symmetry we obtain that \(a_{k,i} = 1/M\), because \(\alpha_{i,m} = 1/M\) for all advertiser \(i\) and auction \(m\). Evaluating at \(\mu = 0\) we obtain that \(\gamma_{k,i} = \gamma := \int_0^\bar{v} \ell(x)^2 (1 - \bar{F}(x)/M)^{K-2}dx\) for all \(k \neq i\) and \(\omega_{k,i} = \omega := \int_0^\bar{v} \bar{F}(x)\ell(x)(1 - \bar{F}(x)/M)^{K-2}dx\) for all \(k \neq i\). Therefore, \(\frac{\partial G_k}{\partial \mu_k}(0) = -\frac{K-1}{M} \gamma\) and \(\frac{\partial G_k}{\partial \mu_i}(0) = \frac{1}{M} (\gamma - \omega)\). The eigenvalues of \(- (J_G(0) + J_G(0)^T)/2\) are \(\nu_1 = \frac{K-1}{M} \omega\) with multiplicity 1 and \(\nu_2 = \frac{K\gamma - \omega}{M}\) with multiplicity \(K - 1\). Assume further that the expected number of players per auction \(\kappa := K/M\) is fixed, which implies that the number of auctions is proportional to the number of players. Because \((1 - \bar{F}(x)/M)^{K-2}\) converges to \(e^{-\kappa\bar{F}(x)}\) as \(K \to \infty\) and the integrands are bounded, Dominated Convergence Theorem implies that

\[
\lim_{K \to \infty} \nu_1 = \kappa \int_0^\bar{v} \bar{F}(x)\ell(x)e^{-\kappa\bar{F}(x)} dx > 0,
\]

35
\[
\lim_{K \to \infty} \nu_2 = \kappa \int_0^\infty \ell(x)^2 e^{-\kappa F(x)} dx > 0.
\]

Hence, there exist \( K \in \mathbb{N} \) and \( \lambda' > 0 \) such that for all \( K \geq K \) the minimum eigenvalue value of \(- (J_G(0) + J_G(0)^T)/2 \) is at least \( \lambda' > 0 \). Because densities are continuous, one obtains that \( J_G(\mu) \) is continuous in \( \mu \). Since the eigenvalues of a matrix are continuous functions of its entries, we conclude that there exists \( \lambda \in (0, \lambda'] \) such that for each \( K \geq K \) there exists a set \( U \subset \mathbb{R}^K \) with \( 0 \in U \) such that \( G \) is \( \lambda \)-strongly monotone over \( U \).

\[\square\]

D  Numerical analysis of convergence under simultaneous learning

We next describe the setup and results of numerical experiments we conducted to demonstrate the convergence of dual multipliers established in Theorem 4 and the convergence in performance established in Theorem 5.

Setup and methodology. We simulate the sample path and payoff achieved when \( K \) bidders with symmetric target expenditure rates \( \rho \) follow adaptive pacing strategies throughout synchronous campaigns of \( T \) periods. (The parametric values we tested are provided below). In each period, each advertiser observed a valuation that was drawn independently across advertisers and time periods from a uniform distribution over \([0,1]\). All the advertisers followed an adaptive pacing strategy tuned by a step size \( \varepsilon \), and an initial dual multiplier \( \mu_1 \). The upper bound on the dual multipliers was set to \( \bar{\mu} = 2/\rho \), guaranteeing \( \bar{\mu} > \bar{v}/\rho \) so the second part of Assumption 2 holds. Recall that \( \mu^* \) is the profile of dual multipliers that solves the complementarity conditions given in (5). We note that under uniform valuations one may obtain analytical solution of the form

\[
\mu^*_k = \left[ \frac{K - 1}{K(K+1)\rho} - 1 \right]^+,
\]

for all \( k \in \{1, \ldots, K\} \). Given a vector of multipliers \( \mu \), recall that one denotes

\[
\Psi_k(\mu) = T \left( E_{\nu} \left[ (v_k - (1 + \mu_k)d_k)^+ \right] + \mu_k \rho k \right)
\]

with \( d_k \) as defined in (4.1). Recalling Equation (4.1), \( \Psi_k(\mu^*) \) is the dual performance of advertiser \( k \) when in each period \( t \) each advertiser \( i \) bids \( b_{i,t} = v_{i,t}/(1 + \mu^*_i) \). Similarly to the above, we note
that under the uniform valuations one may obtain an analytical solutions of the form

$$\Psi_k(\mu^*) = T \left( \frac{1}{K(K+1)} + \left[ \frac{K-1}{K(K+1)} - \rho \right] \right),$$

for all $k \in \{1, \ldots, K\}$. Following a profile of adaptive pacing strategies, we generated a sample path of multipliers’ profiles $\mu_t$ and the corresponding sequences of payoffs $\Pi^A_k$, for all $k = 1, \ldots, K$. Then, the time-average mean squared error of the sample path of dual multipliers is given by

$$MSE = \frac{1}{T} \sum_{t=1}^{T} E \left[ \| \mu_t - \mu^* \|^2 \right],$$

and the average loss relative to the profile $\mu^*$ is given by

$$L = \frac{1}{KT} \sum_{k=1}^{K} (\Psi_k(\mu^*) - \Pi^A_k).$$

We explored all combinations of horizon lengths $T \in \{100, 500, 1000, 5000, 10000, 50000\}$, number of bidders $K \in \{2, 5, 10\}$, and target expenditure rates $\rho \in \{0.1, 0.3, 0.5, 0.7\}$, as well as the adaptive pacing strategy step size $\varepsilon \in \{T^{-1}, T^{-2/3}, T^{-1/2}\}$ and initial selection of dual multipliers $\mu_1 \in \{0.001, 0.01, 0.1, 0.3, 0.5, 0.7\}$. In addition to using the original update rule of the adaptive pacing strategy, that is, $\mu_{k,t+1} = P_{[0,\bar{\mu}_k]}(\mu_{k,t} - \varepsilon_{k,t}(\rho - z_{k,t}))$, we also experimented with the following adjusted update rule:

$$\mu_{k,t+1} = P_{[0,\bar{\mu}_k]} \left( \mu_{k,t} - \varepsilon_{k,t} \left( \rho - \frac{1}{t} \sum_{s=1}^{t} z_{k,s} \right) \right).$$

This update rule is an example of a rule that depends on the entire history. At each time period $t + 1$, the direction of the gradient step at that time depends on the difference between the target expenditure rate $\rho$ and the average expenditure rate up to time $t$. Overall, we explored 2,592 combinations of problem parameters $T$, $K$, and $\rho$, as well as the strategy’s parameters $\varepsilon$ and $\mu_1$. Each instance was replicated 100 times, leading to low mean standard errors.

**Results.** The results of our numerical analysis support the theoretical convergence established in Theorems 4 and 5. The key findings are highlighted through representative examples in Figure 1. The upper-left and upper-right parts of Figure 1 demonstrate the asymptotic convergence of dual multipliers and performance, respectively, for two bidders with target expenditure rate $\rho = 0.1$ and initial multiplier $\mu_1 = 0.001$. This selection of initial dual multipliers illustrates a case where
Figure 1: Convergence under simultaneous adoption of adaptive pacing strategies. (Upper left) The linear relations in the log-log plots illustrate the asymptotic convergence rates of dual multipliers for two symmetric bidders with target expenditure rate $\rho = 0.1$ and initial multiplier $\mu_1 = 0.001$, under different selections of step sizes. (Upper right) Convergence in performance for two bidders. (Lower left) Convergence in dual multipliers for five bidders. (Lower right) Convergence in performance for five bidders.

bidders begin by bidding truthfully, and then learn the extent to which they need to shade bids throughout the campaign (in this case $\mu_k^* = 0.66$ for $k \in \{1, 2\}$). The linear relations in the log-log plots illustrate the convergence rates under different selections of step sizes. Notably, when the step size is $\epsilon = T^{-1}$, the third part of Assumption 3 does not hold, and indeed the profile of dual multipliers and the average payoff do not converge to $\mu^*$ and $\Psi_k(\mu^*)$, respectively. On the other hand, under step size selections of $\epsilon = T^{-2/3}$ and $\epsilon = T^{-1/2}$, Assumption 3 holds. (In Example C.7, Appendix C, we demonstrate a numerical estimation of the monotonicity parameter for two bidders with valuations that are drawn from a uniform distribution over $[0, 1]$, and establish that in such a case $\lambda \sim 0.013$.) Indeed, with both of these step sizes the dual multipliers and the average payoff converge to $\mu^*$ and $\Psi_k(\mu^*)$, respectively. A step size selection of $\epsilon = T^{-1/2}$ led to superior convergence rates of $T^{-1/2}$, for both multipliers and performance. This supports the multiplier convergence rate we established theoretically when $\epsilon = T^{-1/2}$, as well as our conjecture on the
performance convergence rate with the same step size.

The lower parts of Figure 1 provide similar results for the case of 5 bidders (in such a case \( \mu_k^* = 0.33 \) for all \( k \in \{1, \ldots, 5\} \)). Results are consistent across different horizon lengths \( T \), number of bidders \( K \), target expenditure rates \( \rho \), and initial multiplier \( \mu_1 \). Notably, when the initial multiplier selection \( \mu_1 \) is higher than \( \mu_k^* \), all the bidders shade bids aggressively at early stages. This leads to lower initial payments and, in turn, overall performance that is better than the limit performance \( \Psi_k(\mu^*) \). The adjusted step size in (B-5) achieved performance that is very similar to the one of the original update rule of the adaptive pacing strategy. The complete set of results for all the numerical experiments is with the authors and is available upon request.

References


